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ON SOME RESULTS CONNECTED WITH ARTIN CONJECTURE

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Abstract: In the present paper we derive an upper bond for the number of primes p, for which a is a primitive root mod p.
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1. Introduction

Let a be a rational integer and let

$$A = A(a) = \{p \mid p \text{ is a prime number and } a \text{ is a primitive root mod } p\}.$$

In 1927 Artin conjectured that $A(a)$ is infinite, provided that a is neither -1 nor a perfect square. More precisely, denoting by $N_a(x)$ the number of primes $p \leq x$ for which $a \in A$, he conjectured that

$$N_a(x) \sim c(a)\frac{x}{\log x} \qquad (x \to \infty) ,$$

where $c(a)$ is a positive constant. This conjecture has been proved by C. Hooley [2] under the assumption that the Riemann hypothesis holds for fields of the type $Q(\sqrt[k]{a}, \sqrt[k]{1})$. He proved, that

$$N_a(x) = c(a)\frac{x}{\log x} + O\left(\frac{x \log\log x}{\log^2 x}\right) \qquad (x \to \infty) , \tag{1.1}$$

and also determined the value of the constant $c(a)$.
Vinogradow in [3], proved unconditionaly, that

$$N_a(x) \leq c(a)\frac{x}{\log x} + O\left(\frac{x}{\log^{\frac{5}{4}} x}(\log\log x)^2\right) , \tag{1.2}$$

where the constant in O depends on a.

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In the following a denotes an integer, $a \neq -1$ or is not a perfect square, H denotes the largest integer with the property that a is a perfect H-th power, p and q denotes prime numbers, and $M = \prod_{p|a} p$.

In the present paper we prove the following theorem:

Theorem. *If $x \geq \exp M$, then*

$$N_a(x) \leq c(a)\frac{x}{\log x} + O\left(H\frac{x}{\log x}\frac{\log^2 \log\log x}{\log\log x}\right), \tag{1.3}$$

where the constant in O is numerical and $c(a)$ is the same constant as in the estimates (1.1) and (1.2).

2. Let us fix the following notation.
Set

$$B(l,a) = \{p \mid p \equiv 1 (\mathrm{mod}\ l) \text{ and } a^{\frac{p-1}{l}} \equiv 1 (\mathrm{mod}\ p)\},$$

where l is a positive integer,

$$A(\xi,\eta) = \bigcap_{\xi \leq q \leq \eta} (B(1,a) - B(q,a)) \qquad (1 \leq \xi \leq \eta),$$

$$M(x,l,a) = \sum_{\substack{p \leq x \\ p \in B(l,a)}} 1$$

$$N(x,\xi,\eta) = \sum_{\substack{p \leq x \\ p \in A(\xi,\eta)}} 1$$

Note that (see [1]) $A \subseteq A(\xi,\eta)$ for all $\xi \leq \eta$, so that for $x \geq 1$, we have

$$N_a(x) \leq N(x,\xi,\eta) \qquad (1 \leq \xi \leq \eta).$$

In particular, we see that

$$N_a(x) = N(x,1,x) \leq N(x,1,\xi) \tag{2.1}$$

for $1 \leq \xi \leq x$.

3. The proof of the theorem will rest on the following lemmas:

Lemma 3.1. *If* $\xi \geq 1$ *and*

$$S(\xi) = \{l \mid l = 1 \text{ or } l = q_1 q_2 \dots q_r, q_i \text{ distinet primes}, \quad q_j \leq \xi \quad (1 \leq j \leq r)\} ,$$

then for $x \geq 2$ *we have the equality*

$$N(x, 1, \zeta) = \sum_{l \in S(\xi)} \mu(l) M(x, l, a) . \tag{3.1}$$

Proof. With the notation of section 2, we have:

$$N(x, 1, \xi) = \sum_{\substack{p \leq x \\ p \in B(1,a)}} 1 - \sum_{\substack{p \leq x \\ p \in \bigcup\limits_{1 \leq q \leq \xi} B(q,a)}} 1 = \sum_{l \in S(\xi)} \mu(l) \sum_{\substack{p \leq x \\ p \in B(l,a)}} 1 =$$

$$= \sum_{l \in S(\xi)} \mu(l) M(x, l, a) . \qquad \blacksquare$$

Lemma 3.2. *Let* $a = a_1 a_2^2$, a_1 *square - free,* l *square free,*

$$\varepsilon(l) = \begin{cases} 2 & \text{if } 2a_1 \mid l \text{ and } a_1 \equiv 1 (\text{mod } 1) \\ 1 & \text{otherwise,} \end{cases}$$

and let (H, l) *be the greatest common divisor of* H, l *(*H *is determinet in section 1).*

Suppose further that $t \geq 1$, $0 < \alpha \leq 1$, $c_1 > 0$ *is a sufficiently small numerical constant and* $c_2 \geq 0$ *is an arbitrary numerical constant.*
If

$$(l^3 M)^{\varphi(l)} \leq \exp\left(\left(\frac{c_1}{c_2 + 1}\right)^2 \frac{\log^\alpha x}{\log^t \log x}\right) ,$$

then

$$\left| M(x, l, a) - \frac{\varepsilon(l)(H, l)}{l \varphi(l)} \pi(x) \right| \leq c_3 \frac{(H, l)\sqrt{M}}{\prod\limits_{p \mid l} (1 - \frac{1}{p})} \frac{x}{(\log x)^2} + \tag{3.2}$$
$$+ c_4 x \exp(-(1,7c_2 + 1,2)\sqrt{\alpha} \log^{\frac{1-\alpha}{2}} x \log^{\frac{1+t}{2}} \log x) .$$

Proof. The lemma follows from Lemma (5.5) and Corollary (5.1) of [4] and from the following identities:

$$M(x, 2m + 1, -a) = M(x, 2m + 1, a)$$
$$M(x, 2m, -a) =$$
$$= 2M(x, 4m, a^2) + M(x, 2m, a^2) - M(x, 4m, a^4) - M(x, 2m, a) . \qquad \blacksquare$$

4. Proof of Theorem. From Lemma 3.2 for $t = 1$, $\alpha = 1$, $c_2 = 1$ and from (3.1) for $\xi = \frac{1}{3}\log\log x$, we have

$$\begin{aligned} N(x,1,\xi) = &\sum_{l=1}^{\infty} \mu(l)\frac{\varepsilon(l)(H,l)}{l\varphi(l)}\pi(x) + O\left(\sum_{l>\xi}\frac{H}{l\varphi(l)}\pi(x)\right) + \\ &+ O\left(\frac{H\sqrt{M}x}{\log^2 x}\sum_{l\in S(\xi)}\frac{1}{\varphi(l)}\right) + O\left(\frac{x}{\log^{2,9} x}2^{\xi}\right) . \end{aligned} \tag{4.1}$$

From [2] (equality 29) we get

$$\sum_{l=1}^{\infty}\mu(l)\frac{\varepsilon(l)(H,l)}{l\varphi(l)} = c(a) . \tag{4.2}$$

Note that $l < e^{2\xi}$ for all $l \in S(\xi)$, hence we have

$$\sum_{l\in S(\xi)}\frac{\sqrt{M}l}{\varphi(l)} = O\left(\sum_{l\in S(\xi)}\sqrt{M}\log\log l\right) = O(\sqrt{M}2^{\xi}\log 2\xi) = O\left(\log^{\frac{6}{7}} x\right) \tag{4.3}$$

and

$$\begin{aligned} &\sum_{l>\xi}\frac{1}{l\varphi(l)} = O\left(\sum_{l>\xi}\frac{\log\log l}{l^2}\right) = \\ &= O\left(\frac{\log^2\xi}{\xi}\sum_{l>\xi}\frac{\log\log l}{l\log^2 l}\right) = O\left(\frac{(\log\log\log x)^2}{\log\log x}\right) . \end{aligned} \tag{4.4}$$

From (4.1), (4.2), (4.3) and (4.4), we get

$$N(x,1,\xi) = c(a)\frac{x}{\log x} + O\left(H\frac{x}{\log x}\frac{(\log\log\log x)^2}{\log\log x}\right)$$

and Theorem follows from inequality (2.1). ■

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