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To Professor Włodzimierz Staś on his 75th birthday

ON SPECTRAL LARGE SIEVE INEQUALITIES

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Abstract: The spectral large sieve inequality due to H. Iwaniec, that is an estimate for the mean square over a spectral interval of a linear form in the Fourier coefficients of Maass wave forms, is reported by use of a formula of Y. Motohashi and the "hybrid" large sieve inequality. Then the result is generalized to the case, where the coefficients of the linear form may depend on the respective eigenvalue of the hyperbolic Laplacian, and also on a well-spaced set of points.

Keywords: spectral theory, large sieve

1. Introduction

Modern spectral methods in analytic number theory frequently lead to spectral sums of linear forms involving Fourier coefficients of Maass wave forms (for surveys of these topics, see e.g. [4], [5]). By analogy with the well-known character large sieve inequalities, estimates for such spectral averages are customarily called *spectral large sieve inequalities*; in fact, there is common basis for both types of the large sieve, as we shall see below. H. Iwaniec [2] was the first to establish spectral large sieve estimates, on the basis of the fundamental work of N. V. Kuznetsov [16] and motivated by an application to the fourth moment of Riemann's zeta-function over short intervals. His inequalities were related to Maass wave forms for the full modular group, and the case of congruence subgroups was dealt with by Deshouillers and Iwaniec [1]. As another variation of his original results, Iwaniec [3] gave analogous mean value estimates for sums involving Fourier coefficients of holomorphic cusp forms or Eisenstein series.

Our object in this paper is to generalize Iwaniec's large sieve estimates in a different direction, variating the *functional structure* of the sums rather than their coefficients. More precisely, the terms in the sums are allowed to depend explicitly on the respective eigenvalue in a flexible way. Thus the eigenvalue itself may appear in addition to the Fourier coefficients of the corresponding Maass wave

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form. Also, a parameter running over a well-spaced set may occur. The underlying motivation for such generalizations will be explained below.

Before that, we recall Iwaniec's basic inequalities. To formulate these, we introduce some notation (for basic definitions, see [2]-[5], [7], [16], or [21]). The Maass wave form $f_j(z) = f_j(x+yi)$, automorphic under the full modular group and attached to the eigenvalue $\lambda_j = 1/4 + \kappa_j^2$ of the hyperbolic Laplacian, is defined in the upper half plane $y > 0$, where it can be represented by its Fourier series (see [16], eq. (3.38), or [21], Lemma 1.4)

$$f_j(z) = \sqrt{y} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \rho_j(n) K_{i\kappa_j}(2\pi|n|y) e(nx),$$

in the standard notation. Writing $\rho_j(n) = \rho_j(1)t_j(n)$ and $\alpha_j = |\rho_j(1)|^2/\cosh(\pi\kappa_j)$, we may now state Iwaniec's first inequality as follows:

$$\sum_{\kappa_j \le K} \alpha_j \left| \sum_{n\le N} a_n t_j(n) \right|^2 \ll (K^2 + N^{1+\varepsilon})\|\mathbf{a}\|^2, \tag{1.1}$$

where $\mathbf{a} = (a_1, \dots, a_N)$, $\|\mathbf{a}\|$ is the usual norm of this vector, and ε is any fixed small positive number (see Theorem 1 in [2] or Theorem 1 in [3]). A "local" variant of (1.1) reads

$$\sum_{K\le\kappa_j\le K+1} \alpha_j \left| \sum_{n\le N} a_n t_j(n) \right|^2 \ll (K+N)(KN)^{\varepsilon}\|\mathbf{a}\|^2, \tag{1.2}$$

and the inequality (1.3) below covers both (1.1) and (1.2) as special cases.

The inequality (1.2) was pointed out without proof by Iwaniec in [6]. Proofs were given in 1991, independently, by W. Luo [17] and the author (in the first unpublished version of the present paper). Some years later, a simplified approach to the spectral large sieve was opened by Y. Motohashi [20] through his new elegant "elementary" proofs for Kuznetsov's trace and sum formulae. A kind suggestion of Prof. Motohashi in 1995 to the author gave rise to a revision of the above mentioned paper. Though the new proof of the spectral large sieve inequality contained in that revised paper (essentially the present one) is now available in [21] (see Theorem 3.3 therein), we prefer to give a brief account of a slightly less precise version of the argument in sec. 3 below to make the presentation more self-contained. The point of the new method is to reduce the spectral large sieve to the ordinary ("hybrid") large sieve.

Theorem 1.1. *For $1 \le \Delta \le K$, we have*

$$\sum_{K\le\kappa_j\le K+\Delta} \alpha_j \left| \sum_{n\le N} a_n t_j(n) \right|^2 \ll (K\Delta + N)\|\mathbf{a}\|^2(KN)^{\varepsilon}. \tag{1.3}$$

It should be noted that estimates for analogous short interval sums weighted by values of $H_j(s)$, the generating Dirichlet series of $t_j(n)$, have been established by Iwaniec [6] and Motohashi [19], for Re $s = 1/2$ or $s = 1/2$.

Let us discuss briefly some applications of the spectral large sieve. To start from relatively recent work, we used (1.3) in [13], [14] as one of the main tools to estimate in mean the inner product of the square of a cusp form against Maass wave forms.

Next, going back to the roots of the theory, the inequality (1.1) played a vital role in Iwaniec's proof [2] of the estimate

$$\int_T^{T+T^{2/3}} |\zeta(1/2+it)|^4\,dt \ll T^{2/3+\varepsilon}. \tag{1.4}$$

We gave a different proof for this in [8] by a more elementary "transformation method" without appealing to the spectral theory in any way, and extended the same argument subsequently to more general mean value problems for exponential sums and L-functions ([9]–[11]). For instance, one may cope with mean values of the form

$$\int_0^V \Bigg|\sum_{m\asymp M} d(m)g(m,v)e(f(m,v))\Bigg|^2\,dv, \tag{1.5}$$

where the functions f and g satisfy certain regularity conditions, and $m \asymp M$ means that $M \ll m \ll M$. Attempts to apply spectral methods to problems like this or its generalizations led (see [12], [15]) to spectral sums of the type

$$\sum_{\kappa_j\sim K} \alpha_j \Bigg|\sum_{n\le N} a_n t_j(n)\phi_n(\kappa_j)e(\psi_n(\kappa_j))\Bigg|^2, \tag{1.6}$$

where $\sim$ indicates that the summation is restricted to the interval $[K, 2K]$. The presence of the functions ϕ_n and ψ_n of the variable κ_j here is a new problem, especially if the exponential factor is allowed to oscillate rapidly as a function of κ_j.

Generalizing the last mentioned problem still a bit, we let ϕ_n and ψ_n depend even on a parameter y running over a well-spaced set $\{y_r\}$ of real numbers which may be normalized to lie in the interval $[0, 1]$. Also, we let κ_j run over an interval $[K, K+\Delta]$ with $1 \le \Delta \le K$. The basic problem then is estimating the sum

$$\sum_{K\le\kappa_j\le K+\Delta} \alpha_j \Bigg|\sum_{n\le N}\sum_{r=1}^{R} a_{nr} t_j(n)\phi_n(\kappa_j, y_r)e(\psi_n(\kappa_j, y_r))\Bigg|^2. \tag{1.7}$$

A model of the estimate we are looking for is Iwaniec's second main theorem (Theorem 2 in [2]), namely the inequality

$$\sum_{\kappa_j\sim K} \alpha_j \Bigg|\sum_{n\le N} a_n t_j(n)\Bigg|^2 \Bigg|\sum_{r\le R} b_r r^{i\kappa_j}\Bigg|^2 \ll K(K+R)\|\mathbf{a}\|^2\|\mathbf{b}\|^2, \tag{1.8}$$

where $N \ll K$. In the sum over r, we may replace each r be the normalized number $y_r = r/R$, so (1.8) turns out to be is a special case of (1.7).

In the work of Iwaniec, the inequality (1.8) played an important role when the fourth moment (1.4) over a single interval was to be generalized to a system of non-overlapping intervals, and likewise an estimate for the sum (1.7) enables one to deal with an analogous generalization of the problem (1.5) involving a well-spaced set of parameters y_r (see [9], [10]). Our result on the sum (1.7) will be formulated and proved as Theorem 4.1 in sec. 4. Its proof combines aspects of the classical and spectral large sieve. This theorem was recently applied in [15] to mean value problems for Dirichlet L-functions and exponential sums twisted with characters.

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2. Motohashi's formula

The philosophy of Kuznetsov's trace or sum formulae is expressing sums of Kloosterman sums in terms of the spectral theory of the hyperbolic Laplacian, or vice versa. The formula (2.2) below due to Motohashi (see [20], eq. (2.2), or [21], the proof of Lemma 2.4) is of this flavour. Its main advantage is perhaps that the variables are nicely separated on the right. We are using standard notations; in particular, $\sigma_\alpha(n) = \sum_{d|n} d^\alpha$, $\delta_{m,n}$ is the Kronecker symbol, $\int_{(\alpha)}$ means that the integral is taken over the line with real part α, and

$$S(m,n;c) = \sum_{\substack{1\le d\le c\\(c,d)=1}} e\left(\frac{md+n\bar{d}}{c}\right), \qquad d\bar{d}\equiv 1 \pmod c \tag{2.1}$$

is the Kloosterman sum.

Lemma 2.1. *For all real numbers t, we have*

$$\begin{aligned}\sum_{j=1}^{\infty} \frac{\overline{\rho_j(m)}\rho_j(n)}{\cosh(\pi\kappa_j)} p(t,\kappa_j) &+ \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sigma_{2ir}(m)\sigma_{-2ir}(n)}{(m/n)^{ir}|\zeta(1+2ir)|^2} p(t,r)\,dr \\ &= \delta_{m,n}\frac{t}{\pi^2\sinh(\pi t)} + \frac{2t}{\pi\sinh(\pi t)}\sum_{c=1}^{\infty} c^{-1}S(m,n;c)w\left(t,\frac{4\pi\sqrt{mn}}{c}\right),\end{aligned} \tag{2.2}$$

where

$$p(t,r) = \frac{\cosh(\pi r)}{\cosh(\pi(r+t))\cosh(\pi(r-t))},$$

and

$$w(t,x) = \frac{1}{2\pi i}\int_{(\alpha)} \frac{\sin(\pi\eta)}{\pi\eta}\Gamma(\eta+it)\Gamma(\eta-it)\left(\frac{x}{2}\right)^{1-2\eta} d\eta$$

with $0 < \alpha < 1/4$.

Integration over t in (2.2) with a Gaussian weight gives rise to the integral

$$I(\eta) = \sin(\pi\eta) \int_{K-\Delta \log K}^{K+\Delta \log K} \Gamma(\eta + it)\Gamma(\eta - it) \exp\left(- \left(\frac{t-K}{\Delta}\right)^2 \right) dt, \qquad (2.3)$$

where $\eta = \alpha + iu$ with α bounded. This integral is easy to estimate.

Lemma 2.2. *Let* K *be a large positive number,* $\varepsilon > 0$ *a fixed small number,* $\eta = \alpha + iu$, *and suppose that*

$$K^{2\varepsilon} \le \Delta \le K^{1-2\varepsilon}. \qquad (2.4)$$

Then, for fixed $\alpha \in (0, 1)$, *we have*

$$I(\eta) \ll K^{-A} \text{ for } |u| \le K^{1-\varepsilon}\Delta, \qquad (2.5)$$

where A *is any fixed positive number. Further,*

$$I(\eta) \ll \Delta |u|^{2\alpha - 1} \text{ for } |u| \ge K^{1-\varepsilon}\Delta \qquad (2.6)$$

if α *lies in a given finite interval.*

Proof. By Stirling's formula, we have

$$\Gamma(\sigma + it) \ll |t|^{\sigma - 1/2} e^{-(\pi/2)|t|} \text{ for } |t| \to \infty \qquad (2.7)$$

if σ is bounded. This implies immediately the estimate (2.6). Also, looking at the exponential factors, we may verify (2.5) for $|u| \le K - K^{1-\varepsilon}$ in the same way. To deal with the ranges $|u \pm K| \le K^{1-\varepsilon}$, complete the segment of integration in $I(\eta)$ to a rectangular contour with one side on the line Im $t = \mp B$, where B is a sufficiently large positive number. Then, applying (2.7) again, we see that $I(\eta)$ is small.

In the remaining case

$$K + K^{1-\varepsilon} \le |u| \le K^{1-\varepsilon}\Delta,$$

we utilize the oscillatory nature of the integral. By Stirling's formula, the oscillatory factor of the integrand of $I(\eta)$ is

$$\exp(i((u+t)\log(u+t) + (u-t)\log(u-t))) = \exp(i\varphi(t)),$$

say, where

$$|\varphi'(t)| = \left| \log\left(\frac{u+t}{u-t}\right) \right| \gg |K/u| \gg \Delta^{-1}K^{\varepsilon}.$$

Hence, we may verify (2.5) by repeated integration by parts. ■

3. Proof of Theorem 1.1

We suppose that Δ lies in the interval (2.4); it will be clear that this is no essential restriction.

To create the sum on the left of (1.3), we multiply the formula (2.2) by the factor

$$Kt^{-1}\overline{a_m}a_n\sinh(\pi t)\exp\left(-\left(\frac{t-K}{\Delta}\right)^2\right),$$

integrate over $|t-K|\le\Delta\log K$, and finally sum over $m,n\le N$. Then we obtain

$$\begin{aligned}\sum_{K-\Delta\le\kappa_j\le K+\Delta}\alpha_j\left|\sum_{n\le N}a_nt_j(n)\right|^2 &\ll K\Delta\|\mathbf{a}\|^2 \\ &+K\left|\sum_{c=1}^{\infty}\sum_{m,n\le N}\overline{a_m}a_n\int_{(\alpha)}\frac{I(\eta)}{\pi\eta}(2\pi)^{1-2\eta}(mn)^{1/2-\eta}c^{2\eta-2}S(m,n;c)\,d\eta\right|;\end{aligned}\tag{3.1}$$

note that the integral on the left of (2.2) gives a nonnegative contribution which can be omitted in the upper estimation.

By Lemma 2.2, the significant range for η is $|u|\ge K^{1-\varepsilon}\Delta$. In the corresponding two integrals (for positive and negative values of u), we move the integration to the left for all $c>NK^{-1+2\varepsilon}\Delta^{-1}$. Then it turns out, by (2.6), that the contribution of these values of c is small, whence the sum over c can be truncated to

$$c\le NK^{-1+2\varepsilon}\Delta^{-1}.$$

This shows that the second term on the right of (3.1) is negligible if $N<K^{1-2\varepsilon}\Delta$, and the assertion (1.3) is clear in that case. Henceforth we may suppose that N exceeds a power of K, so any factor K^ε may be viewed as N^ε for some other small ε.

Let now $U\ge K^{1-\varepsilon}\Delta$, $C<NK^{-1+2\varepsilon}\Delta^{-1}$, $\alpha=1/2-\varepsilon$, and consider the the second term on the right of (3.1) over $c\sim C$ and $u\sim U$. Insert the Kloosterman sums according to the definition (2.1) and apply Cauchy's inequality to the sum over d. Noting that $I(\eta)\ll\Delta U^{-2\varepsilon}$ by (2.6), we end up with expressions of the type

$$K\Delta C^{-1-2\varepsilon}U^{-1-2\varepsilon}\sum_{c\sim C}\sum_{\substack{1\le d\le c\\(c,d)=1}}\int_U^{2U}\left|\sum_{n\le N}a_nn^{\varepsilon-iu}e\left(\frac{nd}{c}\right)\right|^2du.$$

By the hybrid large sieve inequality (see [21], Lemma 3.11), this is

$$\ll K\Delta C^{-1-2\varepsilon}N^{2\varepsilon}U^{-1-2\varepsilon}(N+C^2U)\|\mathbf{a}\|^2,$$

which is $\ll N^{1+2\varepsilon}K^{2\varepsilon}U^{-2\varepsilon}\|\mathbf{a}\|^2$ by our bounds for C and U. Summing finally over the c- and u-ranges, we complete the proof of the theorem.

4. A generalized large sieve inequality

The following theorem gives a bound for the sum (1.7), and also for the sum (1.6) after an obvious modification for the case when the set $\{y_r\}$ is missing (see Remark 1 below).

Theorem 4.1. *Let $K \geq 1$, $1 \leq \Delta \leq K$, $N \geq 1$, $0 < \delta \leq 1$, and let $y_r \in [0,1]$ for $r = 1, \ldots, R$ be real numbers with $|y_r - y_s| \geq \delta > 0$ for $r \neq s$. Let a_{nr} for $n \sim N$ and $1 \leq r \leq R$ be arbitrary complex numbers, and let $\mathbf{a}$ be the vector composed of these numbers. Let $\phi_n(x,y)$ and $\psi_n(x,y)$ for $n \sim N$ be real continuous functions defined for $K \leq x \leq K + \Delta$ and $0 \leq y \leq 1$. Suppose that*

$$\frac{\partial^j \phi_n}{\partial x^j} \ll \Phi K^{-j} \qquad \text{for } j = 0, 1, 2, \tag{4.1}$$

$$\frac{\partial^j \psi_n}{\partial x^j} \ll \Psi K^{-j} \qquad \text{for } j = 1, 2, \tag{4.2}$$

$$\frac{\partial^2 \psi_n}{\partial x \partial y} \gg \Psi' K^{-1}, \tag{4.3}$$

$$\frac{\partial^3 \psi_n}{\partial x^2 \partial y} \ll \Psi' K^{-2}, \tag{4.4}$$

for certain positive parameters Φ, Ψ, and Ψ', with the assumption that the partial derivatives here are continuous. Define

$$\lambda = \min(\Delta, K/\Psi). \tag{4.5}$$

Then, for any fixed $\varepsilon > 0$, we have

$$\sum_{K \leq \kappa_j \leq K+\Delta} \alpha_j \left| \sum_{n \sim N} \sum_{r=1}^{R} a_{nr} t_j(n) \phi_n(\kappa_j, y_r) e(\psi_n(\kappa_j, y_r)) \right|^2 \\ \ll \Phi^2 \left(K + \frac{K+N}{\lambda} \right) \left(\Delta + K(\Psi'\delta)^{-1} \log(2/\delta) \right) \|\mathbf{a}\|^2 (KN)^{\varepsilon}. \tag{4.6}$$

Proof. Denote the sum on the left of (4.6) by S, put $k_0 = [\Delta/\lambda]$, and let $S(x)$ be the subsum where κ_j is restricted to the interval $[x, x+\lambda)$. Then

$$S = \sum_{k=0}^{k_0} S(K + k\lambda), \tag{4.7}$$

where the last subsum may be incomplete. However, for the uniformity of notation, we write the upper limit of summation in the kth sum throughout as $K + (k+1)\lambda$ with the understanding that for $k = k_0$ this means actually $K + \Delta$.

Next we eliminate the dependence of $S(K+k\lambda)$ on the distribution of the numbers κ_j. To this end, we apply the basic inequality of the large sieve method:

$$|f(u)|^2 \le (b-a)^{-1}\int_a^b |f(x)|^2\,dx + 2\left(\int_a^b |f'(x)|^2\,dx\right)^{1/2}\left(\int_a^b |f(x)|^2\,dx\right)^{1/2}$$
$$\ll (b-a)^{-1}\int_a^b \left(|f(x)|^2 + (b-a)^2|f'(x)|^2\right)\,dx$$

for any $u \in [a,b]$ if f is a continuous function in $[a,b]$ with f' continuous in (a,b) (see [18], Lemma 1.1, applied to f^2). Accordingly, for $K+k\lambda \le \kappa_j \le K+(k+1)\lambda$, we have

$$\left|\sum_{n\sim N}\sum_{r=1}^{R} a_{nr}t_j(n)\phi_n(\kappa_j,y_r)e(\psi_n(\kappa_j,y_r))\right|^2 \ll I_1(k,j)+I_2(k,j), \tag{4.8}$$

where

$$I_1(k,j) = \lambda^{-1}\int_{K+k\lambda}^{K+(k+1)\lambda}\left|\sum_{n\sim N}\sum_{r=1}^{R} a_{nr}t_j(n)\phi_n(x,y_r)e(\psi_n(x,y_r))\right|^2 dx,$$

and $I_2(k,j)$ is a similar expression except that ϕ_n is replaced by the function

$$\tilde{\phi}_n = \left(\frac{\partial\phi_n}{\partial x} + 2\pi i\phi_n\frac{\partial\psi_n}{\partial x}\right)\lambda.$$

In the subsequent treatment of $I_1(k,j)$, we are going to need the assumption (4.1) concerning ϕ_n for $j=0,1$ only, and these conditions hold, by (4.1), (4.2), and (4.5), even for $\tilde{\phi}_n$. Therefore it will suffice to deal with $I_1(k,j)$ alone.

Now, by (4.7) and (4.8),

$$S \ll \sum_{k=0}^{k_0}\sum_{K+k\lambda\le\kappa_j\le K+(k+1)\lambda}\alpha_j(I_1(k,j)+I_2(k,j)) = S_1+S_2,$$

say, and as we pointed out above, it will be suffice to estimate the sum S_1.

The local large sieve inequality (1.3) is applicable to the inner sum in S_1, giving

$$\sum_{K+k\lambda\le\kappa_j\le K+(k+1)\lambda}\alpha_j I_1(k,j) \ll \lambda^{-1}(K(\lambda+1)+N)(KN)^{\varepsilon}$$
$$\times\sum_{n\sim N}\int_{K+k\lambda}^{K+(k+1)\lambda}\left|\sum_{r=1}^{R} a_{nr}\phi_n(x,y_r)e(\psi_n(x,y_r))\right|^2 dx.$$

Then, summing over k, we obtain

$$\begin{aligned} S_1 &\ll \left(K + \frac{K+N}{\lambda}\right)(KN)^{\varepsilon} \sum_{n\sim N} \int_K^{K+\Delta} \left| \sum_{r=1}^{R} a_{nr}\phi_n(x,y_r)e(\psi_n(x,y_r)) \right|^2 dx \\ &= \left(K + \frac{K+N}{\lambda}\right)(KN)^{\varepsilon} \sum_{n\sim N} \sum_{r,s=1}^{R} a_{nr}\bar{a}_{ns} \\ &\quad \times \int_K^{K+\Delta} \phi_n(x,y_r)\phi_n(x,y_s)e(\psi_n(x,y_r) - \psi_n(x,y_s))\, dx. \end{aligned} \tag{4.9}$$

The diagonal terms with $r = s$ give

$$\ll \Delta(K + (K+N)\lambda^{-1})(KN)^{\varepsilon}\Phi^2\|\mathbf{a}\|^2. \tag{4.10}$$

Turning to the nondiagonal terms, note that for $r \neq s$ we have

$$\left| \frac{d}{dx}(\psi_n(x,y_r) - \psi_n(x,y_s)) \right| \gg \Psi' K^{-1}|y_r - y_s|, \tag{4.11}$$

$$\left| \frac{d^2}{dx^2}(\psi_n(x,y_r) - \psi_n(x,y_s)) \right| \ll \Psi' K^{-2}|y_r - y_s| \tag{4.12}$$

by (4.3) and (4.4). Therefore the integral on the right of (4.9) is

$$\ll \Phi^2 K/(\Psi'|y_r - y_s|); \tag{4.13}$$

to see this, write the exponential in the integrand as

$$\frac{e(\psi_n(x,y_r) - \psi_n(x,y_s))\,(\psi_n'(x,y_r) - \psi_n'(x,y_s))}{\psi_n'(x,y_r) - \psi_n'(x,y_s)},$$

and integrate by parts. Consequently, the nondiagonal terms contribute

$$\ll \Phi^2(K/\Psi')\left(K + \frac{K+N}{\lambda}\right)(KN)^{\varepsilon} \sum_{n\sim N} \sum_{r\neq s} |a_{nr}a_{ns}||y_r - y_s|^{-1}.$$

Since $|a_{nr}a_{ns}| \leq \frac{1}{2}(|a_{nr}|^2 + |a_{ns}|^2)$, this is

$$\begin{aligned} &\ll \Phi^2(K/\Psi')\left(K + \frac{K+N}{\lambda}\right)(KN)^{\varepsilon} \sum_{n\sim N} \sum_{r=1}^{R} |a_{nr}|^2 \sum_{s\neq r} |y_r - y_s|^{-1} \\ &\ll \Phi^2 K(\Psi'\delta)^{-1}\log(2/\delta)\left(K + \frac{K+N}{\lambda}\right)(KN)^{\varepsilon}\|\mathbf{a}\|^2. \end{aligned}$$

Combined with (4.10), this completes the proof of the theorem. ■

Remark 1. Suppose that no parameters y_r occur in our sum (so that the functions ϕ_n and ψ_n depend on x only). Then Theorem 4.1 remains valid if all conditions pertaining to y_r, y, and δ are omitted, and the statement of (4.6) is modified as follows:

$$\sum_{K\le\kappa_j\le K+\Delta}\alpha_j\left|\sum_{n\sim N}a_nt_j(n)\phi_n(\kappa_j)e(\psi_n(\kappa_j))\right|^2 \ll \Phi^2\left(K+\frac{K+N}{\lambda}\right)\Delta\|\mathbf{a}\|^2(KN)^{\varepsilon}.$$

Remark 2. Theorem 4.1 contains Iwaniec's estimate (1.8) (up to an unimportant factor) as an immediate corollary. It suffices to show it with the condition $r\sim R$ for the r-sum. Let now $\Delta=K$, $y_r=r/R-1$, $\delta=R^{-1}$, $a_{nr}=a_nb_r$, $\phi_n(x,y)=1$, and $\psi_n(x,y)=(2\pi)^{-1}x\log(y+1)$. Then the assumptions of the theorem are satisfied with $\Phi=1$ and $\Psi=\Psi'=K$, so that $\lambda=1$. We conclude that

$$\sum_{\kappa_j\sim K}\alpha_j\left|\sum_{n\le N}a_nt_j(n)\right|^2\left|\sum_{r\sim R}b_rr^{i\kappa_j}\right|^2 \ll (K+N)(K+R)(KNR)^{\varepsilon}\|\mathbf{a}\|^2\|\mathbf{b}\|^2.$$

Note that this holds independently of the condition $N\ll K$ assumed in (1.8). Also, under this condition, we recover essentially the estimate (1.8).

Remark 3. The condition (4.4) was needed to secure the validity of the estimate (4.13), which is actually a consequence of the familiar "first derivative test" (see [22], Lemma 4.2) if the derivative in (4.11) is monotonic (or has only "few" intervals of monotonicity). Thus, if suitable additional properties of the functions $\psi_n(x,y)$ are known, the condition (4.4) may become redundant.

We take the opportunity to point out that a situation like this occurred in [15], where the condition (4.4) was unfortunately missing when Theorem 4.1 was quotted (as Lemma 5). In that particular application, where extra information (holomorphy in x) on the functions $\psi_n(x,y)$ was available, this omission was of no consequence.

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