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INEQUALITIES FOR THE GRADIENT OF EIGENFUNCTIONS OF THE LAPLACE–BELTRAMI OPERATOR

STEVO STEVIĆ

Abstract: In this paper we shall consider properties of the eigenfunctions of the Laplace-Beltrami operator Δ_ρ and properties of its gradient for a proper domain D with a conformal metric, which density is equal to the reciprocal value of a defining function $\rho(x)$ for this domain i.e. $ds = \rho^{-1}(x)|dx|$.
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1. Introduction

Throughout this paper n is an integer greater than 1, D is a domain in the Euclidean space $\mathbf{R}^n$, $B(a,r) = \{x \in \mathbf{R}^n \| x-a| < r\}$ denotes the open ball centered at a of radius r, where $|x|$ denotes the norm of $x \in \mathbf{R}^n$ and B is the open unit ball in $\mathbf{R}^n$. Let $dV(x)$ denote the Lebesque measure on $\mathbf{R}^n$, $d\sigma$ the surface measure.

We shall say that a locally integrable real valued function f on D possesses the HL-property, with a constant c, if

$$f(a) \leqslant \frac{c}{r^n}\int_{B(a,r)} f(x)dV(x) \quad \text{whenever} \quad B(a,r) \subset D$$

for some $c > 0$ depending only on n.

For example, subharmonic functions possess the HL-property with $c = 1$. In [4] Hardy and Littlewood essentially proved that $|u|^p$, $p > 0$, $n = 2$ also possesses the HL-property whenever u is a harmonic function in D. In the case $n \geqslant 3$ a generalization was made by Fefferman and Stein [3] and Kuran [5]. An elementary proof of this can be found in [7]. In fact the author proved the following theorem:

Theorem A. *If a nonnegative, locally integrable function f possesses the HL-property, with a constant c, then f^p, $p > 0$ also possesses the HL-property but with a different constant c_1 depending only on c, p and n.*

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The following theorem was proved in [8]:

Theorem B. *Let D be a proper subdomain of $\mathbf{R}^n$, $f \in C^2(D)$ such that*

$$|\Delta f(a)| \leqslant \frac{K}{r} \sup_{x \in B(a,r)} |\nabla f(x)| + \frac{K_0}{r^2} \sup_{x \in B(a,r)} |f(x)| \tag{1}$$

where K, K_0 are positive constants independent of $B(a,r) \subset D$. Then $|f|^p$ possesses the HL-property. If (1) holds with $K_0 = 0$, then $|\nabla f|^p$ possesses the HL-property.

A function $\rho(x)$ shall be called (globally) a defining for the domain D if $\rho \in C^1(D_1)$, $\overline{D} \subset D_1$, $d\rho_x \neq 0$, when $x \in \partial D$ and $\rho(x) > 0$, $x \in D$.

The proof of the fact that a defining function exists for every proper domain $D \subset \mathbf{R}^n$ with C^1 boundary can be found in [9]. Observe that this defining function is not unique. For example, if $\rho(x)$ is a defining function then $c\rho(x)$, $c > 0$ is also a defining function for the same domain.

In this paper we shall consider a proper domain D with a conformal metric whose density is equal to the reciprocal value of a defining function for this domain i.e. $ds = \rho^{-1}(x)|dx|$. For such a metric the volume element is $dV_\rho(x) = \rho^{-n}(x)dV(x)$, the surface area element is $d\sigma_\rho(x) = \rho^{1-n}(x)d\sigma(x)$, the normal derivative is $\frac{\partial f}{\partial n_\rho} = \rho(x)\frac{\partial f}{\partial n}$, the gradient is $\nabla_\rho f = \rho(x)\nabla f$, and the Laplace-Beltrami operator is

$$\Delta_\rho f = \rho^n \frac{\partial}{\partial x_i}\left(\rho^{2-n}\frac{\partial f}{\partial x_i}\right) \tag{2}$$

see, for example [1].

In section 2 we shall prove a few auxiliary results.

In section 3 we shall generalize Theorem B and among other results, we shall prove that the eigenfunctions of the Laplace-Beltrami operator Δ_ρ and the norm of its gradient possesses the HL-property, especially the solution to Laplace-Beltrami operator possesses the HL-property. More precisely, we shall prove:

Theorem 1. *If f is an eigenfunction of the Laplace-Beltrami operator Δ_ρ, then $|f|^p$ and $|\nabla f|^p, p > 0$ possesses the HL-property.*

Also we shall give some inequalities for the eigenfunctions and the norm of its gradient. The most important is the following:

Theorem 2. *If f is an eigenfunction of the Laplace-Beltrami operator Δ_ρ, then*

$$\int_D \rho^{\alpha+3p}|\nabla f|^p dV_\rho \leqslant C \int_D \rho^\alpha |f|^p dV_\rho, \quad p > 0, \quad \alpha > 0,$$

where the constant C depends only on D, p, n, λ and α.

One can find some other classes of functions which possess the HL-property in [7], [8] and [10].

2. Preliminaries

One can easily prove the following:

Lemma 1. *Let K be convex compact subset of $\mathbf{R}^n$. If $f \in C^1(K)$, then*
$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x, y \in K)(|x-y| < \delta \Rightarrow |f(x)-f(y)-\langle \nabla f(y), x-y\rangle| \leqslant \varepsilon|x-y|)$.

By Lemma 1 and the Heine-Borel theorem we obtain:

Lemma 2. *Let K be compact connected subset of domain $D \subset \mathbf{R}^n$. If $f \in C^1(D)$, then*
$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x, y \in K)(|x-y| < \delta \Rightarrow |f(x)-f(y)-\langle \nabla f(y), x-y\rangle| \leqslant \varepsilon|x-y|)$.

Lemma 3. *If $\rho(x)$ is a defining function for a proper domain $D \subset \mathbf{R}^n$ then there are $A, B > 0$ such that $Ad(x, \partial D) < \rho(x) < Bd(x, \partial D)$ whenever $x \in D$.*

Proof. For any $x \in D$ there is $x_m \in \partial D$ such that $d(x, x_m) = d(x, \partial D)$.

By Lemma 2

$$|\rho(x) - \rho(x_m) - \langle \nabla\rho(x_m), x - x_m\rangle| < \varepsilon|x - x_m| \quad \text{when} \quad |x - x_m| < \delta.$$

Since $\rho(x_m) = 0$, it follows that

$$|\rho(x)| > |\langle \nabla\rho(x_m), x - x_m\rangle| - \varepsilon|x - x_m|\,, \quad \text{when} \quad |x - x_m| < \delta.$$

On the other hand, the vector $x - x_m$ is orthogonal on the tangential hyperplane of the hypersurface $\rho(x) = 0$ in x_m i.e. $\nabla\rho(x_m)$ and $x - x_m$ are colinear vectors. Therefore

$$|\langle \nabla\rho(x_m), x - x_m\rangle| = |\nabla\rho(x_m)|\; |x - x_m|$$

from which we get

$$|\rho(x)| > (|\nabla\rho(x_m)| - \varepsilon)|x - x_m|, \qquad \text{when} \qquad |x - x_m| < \delta.$$

Since $\rho(x)$ is a defining function then $\nabla\rho(x) \neq 0$, $x \in \partial D$. Consequently from $\rho \in C^1(\overline{D})$ we get that $\min_{x\in\partial D} |\nabla\rho(x)| = m > 0$. For $\varepsilon < m$ choosing $\varepsilon = m/2$ we get $|\rho(x)| > \frac{m}{2}|x-x_m|$ i.e. $\rho(x) > \frac{m}{2}|x-x_m|$ when x is in the δ-neighbourhood of ∂D. The set $D_1 = \{x \in D|\ d(x, \partial D) \geqslant \delta\}$ is compact, therefore $\rho(x)$ has a minimum $M_1 > 0$. In the same manner we can conclude that $d(x, \partial D)$ has a maximum $M_2 > 0$ in D_1. For $c < M_1/M_2$, $c > 0$ we get $\rho(x) > cd(x, \partial D)$, $x \in D_1$. From all of the above we conclude that we can choose $A = \min\left(c, \frac{m}{2}\right)$.

From

$$|\rho(x)| = |\rho(x) - \rho(x_m)| \leqslant |x - x_m| \sup_{t\in[0,1]} |\nabla\rho(x + (x_m - x)t)|$$
$$\leqslant |x - x_m| \sup_{x\in\overline{D}} |\nabla\rho(x)|$$

we can conclude that we can choose $B = \sup_{x\in D} |\nabla\rho(x)|$. B is finite since $\rho \in C^1(\overline{D})$. ■

Hereafter we shall consider that the defining function $\rho(x)$ is a real valued C^2 function.

Then next lemma is a special case of the Green's formula which is valid on Riemannian manifolds.

Lemma 4. *Let $\rho(x)$ be a defining the function of D, and let function $f \in C^2(\overline{D})$. Then*

$$\int_{B(a,r)} \Delta_\rho f dV_\rho = \int_{\partial B(a,r)} \frac{\partial f}{\partial n_\rho} d\sigma_\rho \qquad \text{whenever } \overline{B(a,r)} \subset D.$$

3. Proof of the main results

In this section $\rho(x)$ is a defining function for a proper domain $D \subset \mathbf{R}^n$ with a conformal metric with density equal to the reciprocal value of the defining function for this domain i.e. $ds = \rho^{-1}(x)|dx|$, Δ_ρ is the corresponding Laplace-Beltrami operator for such a metric.

The following three lemmas generalize Theorem B in the case $K_0 = 0$.

Lemma 5. *Let D be a proper subdomain of $\mathbf{R}^n$, $f \in C^2(D)$ such that*

$$|\Delta f(a)| \leqslant \frac{c}{r^k} \sup_{x \in B(a,r)} |\nabla f(x)|$$

for some $c > 0$ and $k \in \mathbf{N}$, whenever $B(a,r) \subset D$. Then

$$|\nabla f(a)| \leqslant \frac{c_1}{r^k} \sup_{x \in B(a,r)} |f(x) - f(a)|,$$

for some $c_1 > 0$, whenever $B(a,r) \subset D$.

Proof. Since D is a proper domain we can suppose that $r \in [0,1]$. Also, it is enough to prove the theorem for closed balls in D.

In [8], the following inequality was proved:

$$|\nabla f(a)| \leqslant \frac{n}{r} \sup_{x \in B(a,r)} |f(x)| + \frac{n}{n+1} r \sup_{x \in B(a,r)} |\Delta f(x)|,$$

whenever $B(a,r) \subset D$ for $f \in C^2(D)$.

By translations we can reduce the proof to the case $a = 0$. Let $\overline{B(0,\rho)} \subset D$ and $M_f = \sup_{B(0,\rho)} |f(x)|$. Choose $\hat{a} \in B(0,\rho)$ so that the function $g(x) = |\nabla f(x)|(\rho - |x|)^k$ attains its maximum at $\hat{a} \in \overline{B(0,\rho)}$. This implies that on the ball $B\left(\hat{a}, \frac{\rho - |\hat{a}|}{2}\right)$ we have:

$$|\nabla f(x)| \leqslant |\nabla f(\hat{a})| \sup_{x \in B\left(\hat{a}, \frac{\rho-|\hat{a}|}{2}\right)} \left(\frac{\rho - |\hat{a}|}{\rho - |x|}\right)^k = 2^k |\nabla f(\hat{a})|.$$

From the hypotheses we have

$$|\nabla f(\hat{a})| \leqslant \frac{n}{r} M_f + \frac{nc}{n+1} \frac{r}{t^k} \sup_{x \in B(\hat{a},s)} |\nabla f(x)|,$$

where $s = r + t$, $r, t > 0$.

Let $s = \frac{\rho - |\hat{a}|}{2}$ and $\frac{nc}{n+1} \frac{r}{t^k} = \frac{1}{2^{k+1}}$. From that we have $\frac{(n+1)}{cn2^{k+1}} t^k + t = \frac{\rho - |\hat{a}|}{2}$. It is easy to see that this equation has a unique positive root t_0 which belongs to the interval $\left(0, \frac{\rho - |\hat{a}|}{2}\right)$. Since $t \in (0,1)$ we have $\left(\frac{(n+1)}{cn2^{k+1}} + 1\right) t > \frac{\rho - |\hat{a}|}{2}$, which implies $L_1 \left(\frac{\rho - |\hat{a}|}{2}\right)^k < r < L_2 \left(\frac{\rho - |\hat{a}|}{2}\right)^k$ for some $L_1, L_2 > 0$. From all of the above we get

$$|\nabla f(\hat{a})| \leqslant \frac{n}{r} M_f + \frac{1}{2^{k+1}} 2^k |\nabla f(\hat{a})| \quad \text{i.e.} \quad |\nabla f(\hat{a})| \leqslant \frac{2nM_f}{r} \leqslant \frac{2^{k+1} n M_f}{L_1 (\rho - |\hat{a}|)^k}.$$

Thus

$$g(0) = |\nabla f(0)| \rho^k \leqslant |\nabla f(\hat{a})| (\rho - |\hat{a}|)^k \leqslant \frac{2^{k+1} n M_f}{L_1} = \frac{2^{k+1} n}{L_1} \sup_{x \in B(0,\rho)} |f(x)|.$$

Applying the above to the function $f(x) - b$, $b \in \mathbf{R}$ and puting $b = f(0)$ we obtain the desired result. ■

Lemma 6. *Let D be a proper subdomain of $\mathbf{R}^n$, $f \in C^{(1)}(D)$ such that*

$$|\nabla f(a)| \leqslant \frac{c}{r^k} \sup_{x \in B(a,r)} |f(x)|,$$

for some $c > 0$ and $k \in \mathbf{N}$, whenever $B(a,r) \subset D$. Then the function $|f|^p$, $(p > 0)$ possesses the HL-property.

Proof. We may assume that $B \subset D$, in contrary we shall consider the function $f(a+rx)$, for $r < d(a, \partial D)$ it is defined on B. Also we may assume that $\int_B |f| = 1$ and $\overline{B} \subset D$.

Let $g(x) = |f(x)|(1 - |x|)^{nk}$. Since $g \in C(\overline{B})$, $g|_{\partial B} \equiv 0$, there is a point $a \in B$ so that the function $g(x)$ attains its maximum i.e. $g(x) \leqslant g(a)$, $x \in B$. By the mean value theorem we have

$$|f(x) - f(a)| \leqslant \sup_{h \in [0,1)} |\nabla f(a + h(x-a))| |x|, \text{ where } x \in B(a,t) \subset B.$$

By the hypotheses we get

$$|f(a)| \leqslant |f(x)| + \frac{tc}{r^k} \sup_{x \in B(a,s)} |f(x)|, \qquad \text{for} \qquad s = t + r, \ x \in B(a,t).$$

Now choose $t, r > 0$ such that $t + r = \frac{1-|a|}{2}$ and $\frac{tc}{r^k} = \frac{1}{2^{nk+1}}$. As in the proof of the previous lemma we can conclude that this system has a unique solution and there are $L_1, L_2 > 0$ such that $L_1(1-|a|)^k < t < L_2(1-|a|)^k$.

On $B\left(a, \frac{1-|a|}{2}\right)$ we have

$$|f(x)| \leqslant \left(\frac{1-|a|}{1-|x|}\right)^{kn} |f(a)| \leqslant \frac{(1-|a|)^{kn}\,|f(a)|}{\left(1-\left|a+\frac{a}{|a|}\frac{1-|a|}{2}\right|\right)^{kn}} = 2^{kn}|f(a)|.$$

Therefore $|f(a)| \leqslant |f(x)| + \frac{1}{2}|f(a)|$, for $x \in B(a,t)$ i.e. $|f(a)| \leqslant 2|f(x)|$. Integrating this inequality over $B(a,t)$ we obtain

$$v_n t^n |f(a)| \leqslant 2\int_{B(a,t)} |f(x)| dV(x) \leqslant 2,$$

which implies

$$|f(a)| \leqslant \frac{2}{v_n t^n} \leqslant \frac{c_1}{(1-|a|)^{kn}}.$$

From that we have $|f(0)| \leqslant c_1 = c_1 \int_B |f| dV$, as desired.

So, the function $|f|$ possesses the HL-property. Thus by Theorem A we obtain that the function $|f|^p$ possesses the HL-property for every $p > 0$. ∎

Lemma 7. *Let D be a proper subdomain of $\mathbf{R}^n$, $f \in C^1(D)$ such that*

$$|\nabla f(a)| \leqslant \frac{c}{r^k} \sup_{x \in B(a,r)} |f(x) - f(a)|$$

for some $c > 0$, and $k \in \mathbf{N}$, whenever $B(a,r) \subset D$. Then $|\nabla f|^p$ $(p > 0)$ possesses the HL-property.

Proof. By Theorem A it is enough to prove that there is a $q > 0$ such that the function $|\nabla f|^q$ possesses the HL-property.

Also it is enough to prove the inequality

$$|\nabla f(0)|^q \leqslant \int_B |\nabla f(x)|^q dV(x).$$

Let $g(x) = f(x) - f(0)$ then

$$|\nabla g(0)| \leqslant \frac{2c}{r^k} \sup_{x \in rB} |g(x)|,$$

where $rB = B(0,r)$.

By Lemma 6, $|g|^p$ possesses the HL-property for every $p > 0$. Thus, we have

$$|\nabla f(0)| = |\nabla g(0)| \leqslant \frac{2^{k+1}c}{r^k} \sup_{x \in \frac{r}{2}B} |g(x)| \leqslant \frac{2^{k+1}c}{r^k} \frac{c_1}{r^n} \int_{rB} |g(x)| dV(x)$$
$$= \frac{c_2}{r^{n+k}} \int_{rB} |g(x)| dV(x).$$

Taking $r = 1$ we obtain

$$|\nabla f(0)| \leqslant c_2 \int_B |g(x)| dV(x) = c_2 \int_B \left| \int_0^1 f'(tx) dt \right| dV(x)$$
$$\leqslant c_2 \int_B \int_0^1 |\nabla f(tx)|\ |x| dt dV(x) = c_2 \int_B |\nabla f(y)| \int_{|y|}^1 \left|\frac{y}{t}\right| dt \frac{1}{t^n} dV(y)$$
$$= c_2 \int_B |\nabla f(y)|\ |y| \frac{|y|^{-n} - 1}{n} dV(y) \leqslant \frac{c_2}{n} \int_B |\nabla f(y)|\ |y|^{1-n} dV(y)$$

since from $y = tx$ we have $0 \leqslant |y| = t|x| < t < 1$. By Hölder's inequality we get

$$|\nabla f(0)| \leqslant \frac{c_2}{n} \left(\int_B |\nabla f(y)|^q dV(y) \right)^{1/q} \left(\int_B |y|^{(1-n)p} dV(y) \right)^{1/p}.$$

Choose $p > 1$, such that the last integral converges. Using polar coordinates we have

$$\int_B |y|^{-(n-1)p} dV(y) = \int_0^1 \int_S \rho^{-(n-1)p} \rho^{n-1} d\sigma(\zeta) d\rho = \frac{1}{(n-1)(1+p)+1},$$

for $\frac{n}{n+1} > p > 1$. For such p we obtain $q = \frac{p}{p-1}$ such that the function $|\nabla f|^q$ possesses the HL-property. ■

We are now in a position to prove Theorem 1.

Proof of Theorem 1. From (2) we have:

$$\Delta_\rho f = \rho^2 (\Delta f - (n-2) \frac{1}{\rho} \langle \nabla \rho, \nabla f \rangle).$$

So, the eigenfunction of the Laplace-Beltrami operator satisfies the partial diffeential equation

$$\Delta f - (n-2) \frac{1}{\rho} \langle \nabla \rho, \nabla f \rangle = \frac{\lambda f}{\rho^2}$$

From this we have

$$|\Delta f| \leqslant \frac{|\lambda|\ |f|}{\rho^2} + \frac{(n-2)}{|\rho|} |\nabla f|\ |\nabla \rho|$$

If $\max_{x\in\overline{D}} |\nabla\rho(x)| = M_\rho$ and A is a constant chosen in a manner described in the proof of the Lemma 3, then

$$|\Delta f(x)| \leqslant \frac{M_\rho |\nabla f(x)|(n-2)}{A\ d(x,\partial D)} + \frac{|\lambda|\ |f(x)|}{A^2\ d(x,\partial D)^2}$$

Thus the eigenfunction satisfies the condition (1). By Theorem B we get that the function $|f|^p, p>0$ possesses the HL-property.

Let us now show that $|\nabla f|^p, p>0$ possesses the HL-property. Let $\overline{B(a,r)} \subset D$, by Lemma 4 and since f is an eigenfunction of the Laplace-Betrami operator we have:

$$\Delta_\rho f(a) \int_{B(a,r)} dV_\rho(x) = -\lambda \int_{B(a,r)} (f(x)-f(a))dV_\rho(x) + \int_{\partial B(a,r)} \frac{\partial f}{\partial n_\rho} d\sigma_\rho.$$

Hence

$$|\Delta_\rho f(a)| \int_{B(a,r)} dV_\rho(x) \leqslant |\lambda| \int_{B(a,r)} |f(x)-f(a)| dV_\rho(x) + \int_{\partial B(a,r)} \left|\frac{\partial f}{\partial n_\rho}\right| d\sigma_\rho.$$

Since

$$\begin{aligned}\int_{B(a,r)} |f(x)-f(a)|dV_\rho(x) &= \int_{B(a,r)} \left|\int_0^1 f'(a+t(x-a))dt\right| dV_\rho(x)\\ &= \int_{B(a,r)} \left|\int_0^1 \langle \nabla f(a+t(x-a)), (x-a)\rangle dt\right| dV_\rho(x)\\ &\leqslant \sup_{x\in B(a,r)} |\nabla f(x)| \int_{B(a,r)} |x-a| dV_\rho(x)\\ &\leqslant r \sup_{x\in B(a,r)} |\nabla f(x)| \int_{B(a,r)} dV_\rho(x)\end{aligned}$$

and

$$\int_{\partial B(a,r)} \left|\frac{\partial f}{\partial n_\rho}\right| d\sigma_\rho \leqslant M_\rho \sup_{x\in B(a,r)} |\nabla f(x)| \int_{\partial B(a,r)} d\sigma_\rho$$

where $M_\rho = \max_{x\in\overline{D}} |\rho(x)|$, we obtain

$$|\Delta_\rho f(a)| \leqslant \sup_{B(a,r)} |\nabla f(x)| \left(r|\lambda| + M_\rho \frac{\int_{\partial B(a,r)} d\sigma_\rho}{\int_{B(a,r)} dV_\rho(x)} \right), \tag{3}$$

whenever $\overline{B(a,r)} \subset D$.

By Lemma 3 we have

$$\frac{\int_{\partial B(a,r)} d\sigma_\rho}{\int_{B(a,r)} dV_\rho(x)} \leqslant C_1 \frac{\int_{\partial B(a,r)} \frac{d\sigma(\xi)}{d(\xi,\partial D)^{n-1}}}{\int_{B(a,r)} \frac{dV(x)}{d(x,\partial D)^n}}, \qquad \text{whenever } \overline{B(a,r)} \subset D.$$

It is clear that $B(a,r/2) \subset B(a,d(a,\partial D)/2)$. If $x \in B(a,d(a,\partial D)/2)$, we can conclude that

$$\frac{1}{2}d(a,\partial D) < d(x,\partial D) < \frac{3}{2}d(a,\partial D). \tag{4}$$

From that we get

$$\frac{\int_{\partial B(a,r/2)} \frac{d\sigma(\xi)}{d(\xi,\partial D)^{n-1}}}{\int_{B(a,r/2)} \frac{dV(x)}{d(x,\partial D)^n}} \leqslant C_2 d(a,\partial D)\frac{\int_{\partial B(a,r/2)} d\sigma(\xi)}{\int_{B(a,r/2)} dV(x)} \leqslant C_3\frac{diam(\overline{D})}{r}. \tag{5}$$

From (3) and (5) we have

$$|\Delta_\rho f(a)| \leqslant \sup_{B(a,r/2)} |\nabla f(x)| \left(\frac{r}{2}|\lambda| + M_\rho C_3\frac{diam(\overline{D})}{r}\right) \leqslant \frac{K}{r} \sup_{B(a,r/2)} |\nabla f(x)|.$$

Thus,

$$|\Delta f(a)| \leqslant \frac{K}{r^3} \sup_{x\in B(a,r)} |\nabla f(x)|$$

whenever $B(a,r) \subset D$.

By Lemma 5 and Lemma 7, we obtain that $|\nabla f(x)|^p, p > 0$ possesses the HL-property.

Lemma 8. *If f is an eigenfunction of the Laplace-Beltrami operator Δ_ρ, then*

$$(r^3|\nabla f(x)|)^p \leqslant \frac{C}{r^n}\int_{B(x,r)} |f|^p dV, p > 0 \tag{6}$$

whenever $B(x,r) \subset D$, where $C = C(p,n,\lambda)$ is a constant.

Proof. By Theorem 1, we have

$$|f(x)|^p \leqslant \frac{C_1}{r^n}\int_{B(x,r)} |f|^p dV, \quad \text{whenever} \quad B(x,r) \subset D.$$

By Lemma 5, we have

$$|\nabla f(x)| \leqslant \frac{K}{r^3} \sup_{y\in B(x,r)} |f(y)|. \tag{7}$$

From (7) we get

$$|\nabla f(x)|^p \leqslant \left(\frac{8K}{r^3} \sup_{y\in B(x,r/2)} |f(y)|\right)^p.$$

Since

$$|f(y)|^p \leqslant \frac{C_1 2^n}{r^n}\int_{B(y,r/2)} |f|^p dV, \qquad y \in B(x,r/2),$$

we have

$$\sup_{y\in B(x,r/2)} |f(y)|^p \leqslant \frac{C_1 2^n}{r^n}\int_{B(x,r)} |f|^p dV,$$

and thus (6) follows. ■

Proof of Theorem 2. Let us put $r = d(a, \partial D)/2$ in (6), we have

$$d(a,\partial D)^{3p}|\nabla f(a)|^p \leqslant \frac{C}{d(a,\partial D)^n}\int_{B(a,d(a,\partial D)/2)}|f(x)|^p dV(x).$$

Since, by Lemma 3 there are $A, B > 0$ such that

$$Ad(a,\partial D) < \rho(a) < Bd(a,\partial D), \tag{8}$$

whenever $a \in D$, we have

$$\rho^{3p}(a)|\nabla f(a)|^p \leqslant \frac{C}{d(a,\partial D)^n}\int_{B(a,d(a,\partial D)/2)}|f(x)|^p dV(x). \tag{9}$$

Multiplying (9) by $\rho^\alpha(a)dV_\rho(a)$ and then integrating over D, we obtain

$$\begin{aligned}&\int_D \rho^{\alpha+3p}(a)|\nabla f(a)|^p dV_\rho(a)\\ \leqslant &C\int_D \frac{\rho^\alpha(a)}{d(a,\partial D)^n}\int_{B(a,d(a,\partial D)/2)}|f(x)|^p dV(x)dV_\rho(a).\end{aligned}$$

By Fubini's theorem we have

$$\begin{aligned}&\int_D \frac{\rho^\alpha(a)}{d(a,\partial D)^n}\int_{B(a,d(a,\partial D)/2)}|f(x)|^p dV(x)dV_\rho(a)\\ =&\int_D |f(x)|^p\int_{E(x)}\frac{\rho^\alpha(a)}{d(a,\partial D)^n}dV_\rho(a)dV(x),\end{aligned}$$

where $E(x) = \{a | x \in B(a, d(a,\partial D)/2)\}$. From (8) we have

$$\begin{aligned}&\int_D |f(x)|^p\int_{E(x)}\frac{\rho^\alpha(a)}{d(a,\partial D)^n}dV_\rho(a)dV(x)\\ \leqslant &C\int_D |f(x)|^p\int_{E(x)} d(a,\partial D)^{\alpha-2n}dV(a)dV(x).\end{aligned}$$

From (4), we obtain

$$\begin{aligned}&\int_D |f(x)|^p\int_{E(x)} d(a,\partial D)^{\alpha-2n}dV(a)dV(x)\\ \leqslant &C\int_D |f(x)|^p d(x,\partial D)^{\alpha-2n}\int_{E(x)} dV(a)dV(x).\end{aligned}$$

Using (8) one more time, we obtain

$$\begin{aligned}&\int_D |f(x)|^p d(x,\partial D)^{\alpha-2n}\int_{E(x)} dV(a)dV(x)\\ \leqslant &C\int_D |f(x)|^p \rho^{\alpha-2n}(x)\int_{E(x)} dV(a)dV(x).\end{aligned}$$

Since $E(x) \subset \{a||a-x| < d(x,\partial D)\}$ we get $\int_{E(x)} dV(a) \leqslant Cd(x,\partial D)^n \leqslant C\rho^n(x)$. Thus

$$\begin{aligned}&\int_D |f(x)|^p \rho^{\alpha-2n}(x) \int_{E(x)} dV(a)dV(x)\\ \leqslant &C\int_D |f(x)|^p \rho^{\alpha-n}(x)dV(x) = C\int_D |f(x)|^p \rho^{\alpha}(x)dV_\rho(x).\end{aligned}$$

From all of the above we obtain the result. ∎

Remark. Throughout the above proof we used C to denote a positive constant which may vary from line to line.

Lemma 9. *If f is an eigenfunction of the Laplace-Beltrami operator Δ_ρ, for $\lambda \neq 0$, then*

$$|f(a)| \leqslant C\left(r + \frac{1}{r|\lambda|}\right) \sup_{x\in B(a,r)} |\nabla f(x)|, \quad \text{whenever } B(a,r) \subset D,$$

where C is a constant depending only on D, λ and n.

Proof. Let $\overline{B(a,r)} \subset D$. By Lemma 4 and since f is an eigenfunction of Laplace-Betrami operator we have

$$\lambda f(a)\int_{B(a,r)} dV_\rho(x) = -\lambda\int_{B(a,r)} (f(x) - f(a))dV_\rho(x) + \int_{\partial B(a,r)} \frac{\partial f}{\partial n_\rho} d\sigma_\rho.$$

If we literarly quote the proof of the second part of Theorem 1 we obtain our result. ∎

Lemma 10. *If f is an eigenfunction of the Laplace-Beltrami operator Δ_ρ, for $\lambda \neq 0$, then*

$$(r|f(a)|)^p \leqslant \frac{C}{r^n}\int_{B(a,r)} |\nabla f(x)|^p dV(x), \tag{10}$$

$p > 0$, whenever $B(a,r) \subset D$, where C is constant depending only on D, p, λ and n.

Proof. By Theorem 1, we get

$$|\nabla f(a)|^p \leqslant \frac{C}{r^n}\int_{B(a,r)} |\nabla f|^p dV, \quad \text{whenever} \quad B(a,r) \subset D.$$

On the other hand, by Lemma 9, we have

$$|f(a)| \leqslant K\left(r + \frac{1}{r|\lambda|}\right) \sup_{x\in B(a,r)} |\nabla f(x)| \tag{11}$$

From (11) we get:

$$|f(a)|^p \leqslant (2K)^p \left(r + \frac{1}{r|\lambda|}\right)^p \left(\sup_{y\in B(a,r/2)} |\nabla f(y)|\right)^p . \tag{12}$$

Since

$$|\nabla f(y)|^p \leqslant \frac{C2^n}{r^n} \int_{B(y,r/2)} |\nabla f|^p dV, \quad y \in B(a,r/2)$$

we have

$$\sup_{y\in B(a,r/2)} |\nabla f(y)|^p \leqslant \frac{C2^n}{r^n} \int_{B(a,r)} |\nabla f|^p dV. \tag{13}$$

Inequality (10) now follows from (12) and (13). ■

By Lemma 10, in the same manner as in Theorem 2, we can prove the following:

Theorem 3. *If f is an eigenfunction of the Laplace-Beltrami operator Δ_ρ, for $\lambda \neq 0$, then*

$$\int_D \rho^{\alpha+p}(x)|f(x)|^p dV_\rho(x) \leqslant C \int_D |\nabla f(x)|^p \rho^\alpha(x) dV_\rho(x), \quad p > 0, \quad \alpha > 0,$$

where C is constant depending only on D, p, n, λ and α.

We leave the proof of this theorem to the reader.

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Address: Matematicki Fakultet, Studentski Trg 16, 11000 Beograd, Serbia
E-mail: `sstevic@ptt.yu; sstevo@matf.bg.ac.yu`
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