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INHOMOGENEOUS DISCRETE CALDERÓN REPRODUCING FORMULAE ASSOCIATED TO PARA-ACCRETIVE FUNCTIONS ON METRIC MEASURE SPACES

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Abstract: Let $(X,\rho,\mu)_{d,\theta}$ be a space of homogeneous type which includes metric measure spaces and some fractals, namely, X is a set, ρ is a quasi-metric on X satisfying that there exist constants $C_0 > 0$ and $\theta \in (0,1]$ such that for all x, x', $y \in X$,

$$|\rho(x,y) - \rho(x',y)| \leqslant C_0\rho(x,x')^\theta[\rho(x,y) + \rho(x',y)]^{1-\theta},$$

and μ is a nonnegative Borel regular measure on X satisfying that for some $d > 0$, all $x \in X$ and all $0 < r < \operatorname{diam} X$,

$$\mu(\{y \in X : \rho(x,y) < r\}) \sim r^d.$$

In this paper, the authors establish the inhomogeneous discrete Calderón reproducing formulae on spaces of homogeneous type associated to a given special para-accretive function introduced by G. David, which will pave the way for developing the theory of Besov and Triebel-Lizorkin spaces on spaces of homogeneous type associated to a given special para-accretive function.
Keywords: space of homogeneous type, para-accretive function, discrete Calderón reproducing formula.

1. Introduction

It is well-known that the remarkable $T1$ theorem given by David and Journé provides a general criterion for the $L^2(\mathbb{R}^n)$-boundedness of generalized Calderón-Zygmund singular integral operators; see [5, 4, 35]. The $T1$ theorem, however, cannot be directly applied to the Cauchy integral on Lipschitz curves. Meyer in [30] (see also [33]) observed that if the function 1 in the $T1$ theorem is allowed to be replaced by a bounded complex-valued function b satisfying $0 < \delta \leqslant \operatorname{Re} b(x)$ almost everywhere, then it would imply the $L^2(\mathbb{R}^n)$ boundedness of the Cauchy

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integral on all Lipschitz curves. Replacing the function 1 by an accretive function b, McIntosh and Meyer in [30] proved the Tb theorem. David, Journé, and Semmes in [6] (see also [4]) introduced a more general class of $L^\infty(\mathbb{R}^n)$ functions b, namely, the so-called para-accretive functions. They proved that the function 1 in the $T1$ theorem can be replaced by a such para-accretive function, which is by now called the Tb theorem. Moreover, they proved that the para-accretivity is also necessary in the sense that the Tb theorem holds for a bounded function b, then b is para-accretive. Meyer in [33] also observed that if $b(x)$ is a bounded function and $1 \leqslant \mathrm{Re}\, b(x)$, one can then define the modified Hardy space $H_b^1(\mathbb{R}^n)$ simply via the classical Hardy space $H^1(\mathbb{R}^n)$, namely, the space $H_b^1(\mathbb{R}^n)$ is defined by the collection of all functions f such that bf is in the classical Hardy space $H^1(\mathbb{R}^n)$. This space has the advantage of the cancellation adapted to the complex measure $b(x)\,dx$ and is closely related to the Tb theorem, where b is an accretive function. In fact, Han, Lee and Lin recently in [17] proved that if $T^*(b) = 0$, then the Calderón-Zygmund operator T is bounded from the classical $H^p(\mathbb{R}^n)$ to a new Hardy space $H_b^p(\mathbb{R}^n)$ for $n/(n+\epsilon) < p \leqslant 1$, where $\epsilon \in (0,1]$ is some positive constant which depends on the regularity of the kernel of the considered Calderón-Zygmund operators. When $p,\ q > 1$, the Besov spaces, $b\dot{B}^s_{pq}(X)$ and $b^{-1}\dot{B}^s_{pq}(X)$, and the Triebel-Lizorkin spaces, $b\dot{F}^s_{pq}(X)$ and $b^{-1}\dot{F}^s_{pq}(X)$, of such type are considered by Han in [14] and the related Tb theorem was also established in that paper. Recently, Deng and the author in [8] complete this theory by establishing the theory of the Besov spaces, $b\dot{B}^s_{pq}(X)$ and $b^{-1}\dot{B}^s_{pq}(X)$, and the Triebel-Lizorkin spaces, $b\dot{F}^s_{pq}(X)$ and $b^{-1}\dot{F}^s_{pq}(X)$, when $p \leqslant 1$ or $q \leqslant 1$. The key tool for developing the theory of such type spaces of functions is the homogeneous continuous or discrete Calderón reproducing formulae; see [14, 17].

The main purpose of this paper is to establish the inhomogeneous discrete Calderón reproducing formulae associated to a given special para-accretive function b introduced by G. David in [4], to pave the way for developing the theory of Besov and Triebel-Lizorkin spaces with $p \leqslant 1$ or $q \leqslant 1$ of such type, which will be considered in another paper; see [18, 19, 20, 22, 23]. The inhomogeneous continuous Calderón reproducing formulae of such type have recently been established in [44], and when $b \equiv 1$, the inhomogeneous discrete Calderón reproducing formulae were obtained in [20]. We point out that due to the inhomogeneity, some new ideas and techniques different from the homogeneous case on $\mathbb{R}^n$ in [14,17] are needed. Moreover, we establish the inhomogeneous discrete Calderón reproducing formulae on spaces of homogeneous type in the sense of Coifman and Weiss in [2, 3], which include metric measure spaces and some fractals. We remark that the analysis on metric spaces has recently obtained an increasing interest; see [34, 25, 13, 27]. Especially, the theory of function spaces on metric spaces, or more generally, the spaces of homogeneous type has been well developed; see [28, 29, 21, 15, 18, 19, 20, 22, 23, 41, 43]. We also point that the spaces of homogeneous type considered in this paper include metric measure spaces, the Euclidean space, the C^∞-compact Riemannian manifolds, the boundaries of Lipschitz domains and, in particular, the Lipschitz manifolds introduced recently by Triebel in [40] and the isotropic and anisotropic d-sets in $\mathbb{R}^n$. It has been proved by Triebel in [38, 39] that the isotropic and anisotropic d-sets in $\mathbb{R}^n$ include various kinds of self-affine fractals, for example, the Cantor set, the generalized Sierpinski carpet and so forth. We particularly point that the spaces of homogeneous type considered in this paper also include the post critically finite self-similar fractals studied by Kigami in [26] and by Strichartz in [34], and the metric spaces with heat kernel studied by Grigor'yan, Hu and Lau in [12]. More examples of spaces of homogeneous type can be found in [2, 3, 34].

To establish the inhomogeneous discrete Calderón reproducing formulae associated to a special para-accretive function b of David on spaces of homogeneous type, we will use Coifman's idea in [6]. That is, based on a given approximation to the identity and the continuous Calderón reproducing formulae in [44], we introduce some kind of discrete Riemann sum operator S (see (3.4) below) and we then verify that S is invertible in the considered space of test functions. To this end, we need to estimate the operator norm of the linear operator $R = I - S$ in the same space of test functions, where I is the identity operator. This will be done by using the related theory of Calderón-Zygmund operators, which needs $R(1) = R^*(b) = 0$. To gúarantee this, we need to make some special choices for the inhomogeneous terms in our discrete Riemann sum operator S.

We remark that using the inhomogeneous discrete Calderón reproducing formulae associated to a special para-accretive function in this paper, we can further establish the inhomogeneous Plancherel-Pôlya inequalities as in [7]. Based on this, we can then develop the theory of Besov and Triebel-Lizorkin spaces associated to a special para-accretive function as in [23], 22]. The details will be presented in another paper.

In the next section, we will recall some definitions and notation, especially, the related theory of Calderón-Zygmund operators and the continuous Calderón reproducing formulae. The discrete Calderón reproducing formulae will be established in Section 3.

2. Preliminaries

A quasi-metric ρ on a set X is a function $\rho: X \times X \to [0, \infty)$ satisfying that

(i) $\rho(x, y) = 0$ if and only if $x = y$;

(ii) $\rho(x, y) = \rho(y, x)$ for all $x, y \in X$;

(iii) There exists a constant $A \in [1, \infty)$ such that for all x, y and $z \in X$,

$$\rho(x, y) \leqslant A[\rho(x, z) + \rho(z, y)].$$

Any quasi-metric defines a topology, for which the balls

$$B(x, r) = \{y \in X : \rho(y, x) < r\}$$

for all $x \in X$ and all $r > 0$ form a basis.

In what follows, we set $\operatorname{diam} X = \sup\{\rho(x, y) : x, y \in X\}$. We also make the following conventions. We denote by $f \sim g$ that there is a constant $C > 0$ independent of the main parameters such that $C^{-1}g < f < Cg$. Throughout the paper, we will denote by C a positive constant which is independent of the main parameters, but it may vary from line to line. Constants with subscripts, such as C_1, do not change in different occurrences. For any $q \in [1, \infty]$, we denote by q' its conjugate index, namely, $1/q + 1/q' = 1$. Let A be a set and we will denote by χ_A the characteristic function of A.

Definition 2.1. ([22]) Let $d > 0$ and $\theta \in (0, 1]$. A space of homogeneous type, $(X, \rho, \mu)_{d,\theta}$, is a set X together with a quasi-metric ρ and a nonnegative Borel regular measure μ on X, and there exists a constant $C_0 > 0$ such that for all $0 < r < \operatorname{diam} X$ and all $x, x', y \in X$,

$$\mu(B(x, r)) \sim r^d \tag{2.1}$$

and

$$|\rho(x, y) - \rho(x', y)| \leqslant C_0 \rho(x, x')^\theta [\rho(x, y) + \rho(x', y)]^{1-\theta}. \tag{2.2}$$

The space of homogeneous type defined above is a variant of the space of homogeneous type introduced by Coifman and Weiss in [2]. In [28], Macias and Segovia have proved that one can replace the quasi-metric ρ of the space of homogeneous type in the sense of Coifman and Weiss by another quasi-metric $\bar{\rho}$ which yields the same topology on X as ρ such that $(X, \bar{\rho}, \mu)$ is the space defined by Definition 2.1 with $d = 1$.

The following construction given by Christ in [1] provides an analogue of the grid of Euclidean dyadic cubes on spaces of homogeneous type.

Lemma 2.1. *Let X be a space of homogeneous type. Then there exists a collection*

$$\{Q_\alpha^k \subset X : k \in \mathbb{Z}_+, \alpha \in I_k\}$$

of open subsets, where I_k is some index set, and constants $\delta \in (0, 1)$ and $C_1, C_2 > 0$ such that

(i) $\mu(X \setminus \bigcup_\alpha Q_\alpha^k) = 0$ *for each fixed k and $Q_\alpha^k \cap Q_\beta^k = \emptyset$ if $\alpha \neq \beta$;*
(ii) *for any α, β, k, l with $l \geqslant k$, either $Q_\beta^l \subset Q_\alpha^k$ or $Q_\beta^l \cap Q_\alpha^k = \emptyset$;*
(iii) *for each (k, α) and each $l < k$ there is a unique β such that $Q_\alpha^k \subset Q_\beta^l$;*
(iv) $\operatorname{diam}(Q_\alpha^k) \leqslant C_1 \delta^k$;
(v) *each Q_α^k contains some ball $B(z_\alpha^k, C_2\delta^k)$, where $z_\alpha^k \in X$.*

In fact, we can think of Q_α^k as being a dyadic cube with diameter rough δ^k and centered at z_α^k. In what follows, we always suppose $\delta = 1/2$; see [21, pp. 96-98] for how to remove this restriction. Also, in the following, for $k \in \mathbb{Z}_+$ and $\tau \in I_k$, we denote by $Q_\tau^{k,\nu}$, $\nu = 1, 2, \cdots, N(k, \tau)$, the set of all cubes $Q_{\tau'}^{k+j} \subset Q_\tau^k$, where

j is a positive large integer whose value will be determined later. Denote by $y_\tau^{k,\nu}$ a point in $Q_\tau^{k,\nu}$. For any dyadic cube Q and any $f \in L^1_{\rm loc}(X)$, we set

$$m_Q(f) = \frac{1}{\mu(Q)} \int_Q f(x)\, d\mu(x).$$

In the sequel, we let

$$\mathbb{J} = \{Q_\tau^{k,\nu} : \ k \in \mathbb{Z}_+, \ \tau \in I_k, \ \nu = 1, \cdots, N(k,\tau)\},$$

the set of all dyadic cubes on X.

Let us now recall the definitions of the para-accretive functions (see [6, 4]) and the space of test functions (see [14]).

Definition 2.2. function on X, a space of homogeneous type.

(i) b is said to be para-accretive if there exist constants $C_3 > 0$ and $\kappa \in (0,1]$ such that for all balls $B \subset X$, there is a ball $B' \subset B$ with $\kappa\mu(B) \leqslant \mu(B')$ satisfying

$$\frac{1}{\mu(B)} \left| \int_{B'} b(x)\, d\mu(x) \right| \geqslant C_3 > 0.$$

(ii) b is said to be special para-accretive if there exists constant $C_4 > 0$ such that for any dyadic cube $Q \in \mathbb{J}$,

$$\frac{1}{\mu(Q)} \left| \int_Q b(x)\, d\mu(x) \right| \geqslant C_4 > 0.$$

In this case, we simply write $b \in SPF(X)$.

Obviously, a special para-accretive function is also a para-accretive function.

Definition 2.3. Fix $\gamma > 0$ and $\theta \geqslant \beta > 0$. A function f defined on X is said to be a test function of type (x_0, r, β, γ) with $x_0 \in X$ and $r > 0$, if f satisfies the following conditions:

(i) $|f(x)| \leqslant C \frac{r^\gamma}{(r+\rho(x,x_0))^{d+\gamma}}$;

(ii) $|f(x)-f(y)| \leqslant C \left(\frac{\rho(x,y)}{r+\rho(x,x_0)}\right)^\beta \frac{r^\gamma}{(r+\rho(x,x_0))^{d+\gamma}}$ for $\rho(x,y) \leqslant \frac{1}{2A}[r+\rho(x,x_0)]$.

If f is a test function of type (x_0, r, β, γ) related to a para-accretive function b, we write $f \in \mathcal{G}(x_0, r, \beta, \gamma)$, and the norm of f in $\mathcal{G}(x_0, r, \beta, \gamma)$ is defined by

$$\|f\|_{\mathcal{G}(x_0,r,\beta,\gamma)} = \inf\{C : \text{(i) and (ii) hold}\}.$$

Now fix $x_0 \in X$ and let $\mathcal{G}(\beta,\gamma) = \mathcal{G}(x_0, 1, \beta, \gamma)$. It is easy to see that

$$\mathcal{G}(x_1, r, \beta, \gamma) = \mathcal{G}(\beta, \gamma)$$

with an equivalent norm for all $x_1 \in X$ and $r > 0$. Furthermore, it is easy to check that $\mathcal{G}(\beta, \gamma)$ is a Banach space with respect to the norm in $\mathcal{G}(\beta, \gamma)$. Also, let the dual space $(\mathcal{G}(\beta, \gamma))'$ be all linear functionals $\mathcal{L}$ from $\mathcal{G}(\beta, \gamma)$ to $\mathbb{C}$ with the property that there exists $C \geqslant 0$ such that for all $f \in \mathcal{G}(\beta, \gamma)$,

$$|\mathcal{L}(f)| \leqslant C\|f\|_{\mathcal{G}(\beta,\gamma)}.$$

We denote by $\langle h, f\rangle$ the natural pairing of elements $h \in (\mathcal{G}(\beta, \gamma))'$ and $f \in \mathcal{G}(\beta, \gamma)$. Clearly, for all $h \in (\mathcal{G}(\beta, \gamma))'$, $\langle h, f\rangle$ is well defined for all $f \in \mathcal{G}(x_0, r, \beta, \gamma)$ with $x_0 \in X$ and $r > 0$.

It is well-known that even when $X = \mathbb{R}^n$, $\mathcal{G}(\beta_1, \gamma)$ is not dense in $\mathcal{G}(\beta_2, \gamma)$ if $\beta_1 > \beta_2$, which will bring us some inconvenience. To overcome this defect, in what follows, for a given $\epsilon \in (0, \theta]$, we let $\mathring{\mathcal{G}}(\beta, \gamma)$ be the completion of the space $\mathcal{G}(\epsilon, \epsilon)$ in $\mathcal{G}(\beta, \gamma)$ when $0 < \beta,\ \gamma < \epsilon$.

Let b be a para-accretive function. As usual, we write

$$b\mathcal{G}(\beta, \gamma) = \{f : \ f = bg \text{ for some } g \in \mathcal{G}(\beta, \gamma)\}.$$

If $f \in b\mathcal{G}(\beta, \gamma)$ and $f = bg$ for some $g \in \mathcal{G}(\beta, \gamma)$, then the norm of f is defined by

$$\|f\|_{b\mathcal{G}(\beta,\gamma)} = \|g\|_{\mathcal{G}(\beta,\gamma)}.$$

By this definition, it is easy to see that

$$f \in \left(b\mathring{\mathcal{G}}(\beta, \gamma)\right)' \quad \text{if and only if} \quad bf \in \left(\mathring{\mathcal{G}}(\beta, \gamma)\right)', \tag{2.3}$$

where we define $bf \in \left(\mathring{\mathcal{G}}(\beta, \gamma)\right)'$ by

$$\langle bf, g\rangle = \langle f, bg\rangle$$

for all $g \in \mathring{\mathcal{G}}(\beta, \gamma)$.

In what follows, we also let

$$\mathcal{G}_0(x_0, r, \beta, \gamma) = \left\{f \in \mathcal{G}(x_0, r, \beta, \gamma) : \ \int_X f(x)b(x)\, d\mu(x) = 0\right\};$$

for $\eta \in (0, \theta]$, we define $C_0^\eta(X)$ be the set of all functions having compact support such that

$$\|f\|_{C_0^\eta(X)} = \sup_{x \neq y} \frac{|f(x) - f(y)|}{\rho(x, y)^\eta} < \infty.$$

Endow $C_0^\eta(X)$ with the natural topology and let $(C_0^\eta(X))'$ be its dual space.

Definition 2.4. Let $\epsilon \in (0, \theta]$. A continuous complex-valued function $K(x, y)$ on

$$\Omega = \{(x, y) \in X \times X : x \neq y\}$$

is called a Calderón-Zygmund kernel of type ϵ if there exists a constant $C_5 > 0$ such that

(i) $|K(x, y)| \leqslant C_5 \rho(x, y)^{-d}$,

(ii) $|K(x, y) - K(x', y)| \leqslant C_5 \rho(x, x')^{\epsilon} \rho(x, y)^{-d-\epsilon}$ for $\rho(x, x') \leqslant \frac{\rho(x,y)}{2A}$,

(iii) $|K(x, y) - K(x, y')| \leqslant C_5 \rho(y, y')^{\epsilon} \rho(x, y)^{-d-\epsilon}$ for $\rho(y, y') \leqslant \frac{\rho(x,y)}{2A}$.

A continuous linear operator $T : C_0^{\eta}(X) \to (C_0^{\eta}(X))'$ for all $\eta \in (0, \theta]$ is a Calderón-Zygmund singular integral operator of type ϵ if there is a Calderón-Zygmund kernel $K(x, y)$ of the type ϵ as above such that

$$\langle Tf, g \rangle = \int_X \int_X K(x, y) f(y) g(x) \, d\mu(x) \, d\mu(y)$$

for all f, $g \in C_0^{\eta}(X)$ with disjoint supports.

We also need the following notion of the strong weak boundedness property in [21, 16].

Definition 2.5. A Calderón-Zygmund singular integral operator T of the kernel K is said to have the strong weak boundedness property, if there exist $\eta \in (0, \theta]$ and constant $C_6 > 0$ such that

$$|\langle K, f \rangle| \leqslant C_6 r^d$$

for all $r > 0$ and all continuous f on $X \times X$ with $\operatorname{supp} f \subseteq B(x_1, r) \times B(y_1, r)$, where x_1 and $y_1 \in X$, $\|f\|_{L^\infty(X \times X)} \leqslant 1$, $\|f(\cdot, y)\|_{C_0^{\eta}(X)} \leqslant r^{-\eta}$ for all $y \in X$ and $\|f(x, \cdot)\|_{C_0^{\eta}(X)} \leqslant r^{-\eta}$ for all $x \in X$. We denote this by $T \in SWBP$.

In what follows, we use $\mathcal{M}_b$ to denote the multiplication operator defined by b, namely, for suitable functions f, $\mathcal{M}_b(f) = bf$. The following lemma when $b = 1$ was established in [16] and when b is a general para-accretive function, it was established in [44].

Lemma 2.2. *Let b be a para-accretive function as in Definition 2.2 and $\epsilon \in (0, \theta]$. Let T be a continuous linear operator from $C_0^{\eta}(X)$ to $(C_0^{\eta}(X))'$ for all $\eta \in (0, \theta]$ such that the kernels of T and $b^{-1}T^*\mathcal{M}_b$ respectively satisfy the conditions (i) and (ii) and only the condition (ii) of Definition 2.4 with the regularity exponent ϵ, $T(1) = 0$, and $T \in SWBP$. Furthermore, $K(x, y)$, the kernel of T, satisfies the following smoothness condition that*

$$\left| \left[K(x, y) b^{-1}(y) - K(x', y) b^{-1}(y)\right] - \left[K(x, y') b^{-1}(y') - K(x', y') b^{-1}(y')\right] \right| \leqslant C \rho(x, x')^{\epsilon} \rho(y, y')^{\epsilon} \rho(x, y)^{-d-2\epsilon} \tag{2.4}$$

for all x, x', y, $y' \in X$ such that $\rho(x, x')$, $\rho(y, y') \leqslant \frac{\rho(x,y)}{3A^2}$. Then for any $x_0 \in X$, $r > 0$ and $0 < \beta$, $\gamma < \epsilon$, T maps $\mathcal{G}_0(x_0, r, \beta, \gamma)$ into itself. Moreover, if we denote by $\|T\|$ the smallest constant in the estimates of the kernel of T, then there exists a constant $C > 0$ such that

$$\|Tf\|_{\mathcal{G}(x_0, r, \beta, \gamma)} \leqslant C \|T\| \|f\|_{\mathcal{G}(x_0, r, \beta, \gamma)}.$$

We now recall the definition of approximations to the identity in [14].

Definition 2.6. Let b be a para-accretive function. A sequence $\{S_k\}_{k\in\mathbb{Z}_+}$ of linear operators is said to be an approximation to the identity of order $\epsilon \in (0,\theta]$ associated to b if there exists $C_7 > 0$ such that for all $k \in \mathbb{Z}$ and all x, x', y and $y' \in X$, $S_k(x,y)$, the kernel of S_k is a function from $X \times X$ into $\mathbb{C}$ satisfying

(i) $|S_k(x,y)| \leqslant C_7 \frac{2^{-k\epsilon}}{(2^{-k}+\rho(x,y))^{d+\epsilon}}$;

(ii) $|S_k(x,y) - S_k(x',y)| \leqslant C_7 \left(\frac{\rho(x,x')}{2^{-k}+\rho(x,y)}\right)^\epsilon \frac{2^{-k\epsilon}}{(2^{-k}+\rho(x,y))^{d+\epsilon}}$ for $\rho(x,x') \leqslant \frac{1}{2A}(2^{-k}+\rho(x,y))$;

(iii) $|S_k(x,y) - S_k(x,y')| \leqslant C_7 \left(\frac{\rho(y,y')}{2^{-k}+\rho(x,y)}\right)^\epsilon \frac{2^{-k\epsilon}}{(2^{-k}+\rho(x,y))^{d+\epsilon}}$ for $\rho(y,y') \leqslant \frac{1}{2A}(2^{-k}+\rho(x,y))$;

(iv) $|[S_k(x,y)-S_k(x,y')]-[S_k(x',y)-S_k(x',y')]| \leqslant C_7 \left(\frac{\rho(x,x')}{2^{-k}+\rho(x,y)}\right)^\epsilon \left(\frac{\rho(y,y')}{2^{-k}+\rho(x,y)}\right)^\epsilon \times \frac{2^{-k\epsilon}}{(2^{-k}+\rho(x,y))^{d+\epsilon}}$ for $\rho(x,x') \leqslant \frac{1}{2A}(2^{-k}+\rho(x,y))$ and $\rho(y,y') \leqslant \frac{1}{2A}(2^{-k}+\rho(x,y))$;

(v) $\int_X S_k(x,y)b(y)\,d\mu(y) = 1$;

(vi) $\int_X S_k(x,y)b(x)\,d\mu(x) = 1$.

Remark 2.1. By Coifman's construction in [6], if b is a given para-accretive function, one can construct an approximation to the identity of order θ such that $S_k(x,y)$ has a compact support when one variable is fixed, namely, there is a constant $C_8 > 0$ such that for all $k \in \mathbb{Z}$, $S_k(x,y) = 0$ if $\rho(x,y) \geqslant C_8 2^{-k}$.

Remark 2.2. We also remark that in the sequel, if the approximation to the identity as in Definition 2.6 exists, then all the results still hold when b and b^{-1} are bounded. It seems that we do not need to assume that b is a para-accretive function. However, in [6], it was proved that the existence of the approximation to the identity as in Definition 2.6 is equivalent to the para-accretivity of b.

The continuous Calderón reproducing formulae associated to a given para-accretive function were established in [44].

Lemma 2.3. *Let b be a para-accretive function, $\epsilon \in (0,\theta]$, $\{S_k\}_{k=0}^\infty$ be an approximation to the identity of order ϵ. Set $D_k = S_k - S_{k-1}$ for $k \in \mathbb{N}$ and $D_0 = S_0$. Then there exists a family of linear operators $\widetilde{D}_k$ for $k \in \mathbb{Z}_+$ and a fixed large integer $N \in \mathbb{N}$ such that for all $f \in \mathcal{G}(\beta,\gamma)$ with $0 < \beta$, $\gamma < \epsilon$,*

$$f = \sum_{k=0}^{\infty} \widetilde{D}_k \mathcal{M}_b D_k \mathcal{M}_b(f) = \sum_{k=0}^{\infty} D_k \mathcal{M}_b \widetilde{E}_k \mathcal{M}_b(f), \tag{2.5}$$

where the series converge in the norm of $\mathcal{G}(\beta',\gamma')$ for $0 < \beta' < \beta$ and $0 < \gamma' < \gamma$. Moreover, (2.5) also converge in the norm of $L^p(X)$ for $p \in (1,\infty)$, and the kernels of the operators $\widetilde{D}_k$ satisfy the conditions (i) and (ii) of Definition 2.6 with ϵ replaced by ϵ' for $0 < \epsilon' < \epsilon$, and

$$\int_X \widetilde{D}_k(x,y)b(y)\,d\mu(y) = \int_X \widetilde{D}_k(x,y)b(y)\,d\mu(x) = \begin{cases} 1, & k = 0,1,\cdots,N, \\ 0, & k \geqslant N+1; \end{cases} \tag{2.6}$$

the kernels of the operators $\widetilde{E}_k$ satisfy the conditions (i) and (iii) of Definition 2.6 with ϵ replaced by ϵ' for $0 < \epsilon' < \epsilon$ and (2.6).

Lemma 2.4. *With all the notation as in Lemma 2.3, then for all $f \in b\mathcal{G}(\beta, \gamma)$,*

$$f = \sum_{k=0}^{\infty} \mathcal{M}_b \widetilde{D}_k \mathcal{M}_b D_k(f) = \sum_{k=0}^{\infty} \mathcal{M}_b D_k \mathcal{M}_b \widetilde{E}_k(f)$$

holds in both the norm of $b\mathcal{G}(\beta', \gamma')$ for $0 < \beta' < \beta$ and $0 < \gamma' < \gamma$, and the norm of $L^p(X)$ with $p \in (1, \infty)$.

Lemma 2.5. *With all the notation as in Lemma 2.3, then for all $f \in \left(\mathring{\mathcal{G}}(\beta, \gamma)\right)'$ with $0 < \beta,\ \gamma < \epsilon$,*

$$f = \sum_{k=0}^{\infty} \mathcal{M}_b D_k \mathcal{M}_b \widetilde{E}_k(f) = \sum_{k=0}^{\infty} \mathcal{M}_b \widetilde{D}_k \mathcal{M}_b D_k(f)$$

holds in $\left(\mathring{\mathcal{G}}(\beta', \gamma')\right)'$ with $\beta < \beta' < \epsilon$ and $\gamma < \gamma' < \epsilon$.

Lemma 2.6. *With all the notation as in Lemma 2.3, then for all $f \in \left(b\mathring{\mathcal{G}}(\beta, \gamma)\right)'$ with $0 < \beta,\ \gamma < \epsilon$,*

$$f = \sum_{k=0}^{\infty} \widetilde{D}_k \mathcal{M}_b D_k \mathcal{M}_b(f) = \sum_{k=0}^{\infty} D_k \mathcal{M}_b \widetilde{E}_k \mathcal{M}_b(f)$$

holds in $\left(b\mathring{\mathcal{G}}(\beta', \gamma')\right)'$ with $\beta < \beta' < \epsilon$ and $\gamma < \gamma' < \epsilon$.

Let b be a para-accretive function and $\{S_k\}_{k \in \mathbb{Z}_+}$ be an approximation to the identity associated to b as in Definition 2.6. The Littlewood-Paley g-function is defined by

$$g(f)(x) = \left[\sum_{k=0}^{\infty} |D_k(f)(x)|^2\right]^{1/2}, \tag{2.7}$$

where $D_0 = S_0$ and $D_k = S_k - S_{k-1}$ for $k \in \mathbb{N}$. In [44], the following Littlewood-Paley g-function was established via the above continuous Calderón reproducing formulae.

Lemma 2.7. *Let b be a para-accretive function and $\{S_k\}_{k=0}^{\infty}$ be an approximation to the identity of order $\epsilon \in (0, \theta]$ as in Definition 2.6. Let $\{D_k\}_{k \in \mathbb{Z}_+}$ be as above and $g(f)$ be defined as in (2.7). Then for any $p \in (1, \infty)$, there exist two constants A_p and B_p depending on p such that for all $f \in L^p(X)$,*

$$A_p \|f\|_{L^p(X)} \leqslant \|g(f)\|_{L^p(X)} \leqslant B_p \|f\|_{L^p(X)}.$$

We also need the following Fefferman-Stein vector-valued maximal function inequality in [9].

Lemma 2.8. *Let $1 < p < \infty$, $1 < q \leqslant \infty$ and M be the Hardy-Littlewood maximal operator on X. Let $\{f_k\}_{k=0}^{\infty} \subset L^p(X)$ be a sequence of measurable functions on X. Then*

$$\left\| \left\{ \sum_{k=0}^{\infty} |M(f_k)|^q \right\}^{1/q} \right\|_{L^p(X)} \leqslant C \left\| \left\{ \sum_{k=0}^{\infty} |f_k|^q \right\}^{1/q} \right\|_{L^p(X)},$$

where C is independent of $\{f_k\}_{k=0}^{\infty}$.

3. Discrete Calderón reproducing formulae

In this section, we establish the discrete Calderón reproducing formulae. Let b be a special para-accretive function of David, $\{S_k\}_{k\in\mathbb{Z}_+}$ be an approximation to the identity of order $\epsilon \in (0, \theta]$ as in Definition 2.6, $\{D_k\}_{k\in\mathbb{Z}_+}$ be as in Section 2 and $D_k = 0$ for $k = -1, -2, \cdots$. In what follows, for any $Q \in \mathbb{J}$, we set

$$b(Q) = \int_Q b(x)\, d\mu(x). \tag{3.1}$$

For $k \in \mathbb{Z}_+$ and $N \in \mathbb{N}$, let

$$D_k^N = \sum_{|l|\leqslant N} D_{k+l}. \tag{3.2}$$

We now introduce the following discrete Riemann sum operator

$$\begin{aligned} S(f)(x) = & \sum_{k=0}^{N} \sum_{\tau\in I_k} \sum_{\nu=1}^{N(k,\tau)} \int_{Q_\tau^{k,\nu}} D_k^N(x,y) b(y)\, d\mu(y) \\ & \times \left\{ \frac{1}{b(Q_\tau^{k,\nu})} \int_{Q_\tau^{k,\nu}} b(u) D_k \mathcal{M}_b(f)(u)\, d\mu(u) \right\} \\ & + \sum_{k=N+1}^{\infty} \sum_{\tau\in I_k} \sum_{\nu=1}^{N(k,\tau)} \int_{Q_\tau^{k,\nu}} D_k^N(x,y) b(y)\, d\mu(y) D_k \mathcal{M}_b(f)(y_\tau^{k,\nu}), \end{aligned}$$

where $y_\tau^{k,\nu}$ with $k \in \mathbb{N}$, $\tau \in I_k$ and $\nu = 1, \cdots, N(k,\tau)$ can be any fixed point in $Q_\tau^{k,\nu}$. We need these special choices to guarantee $S(1) = 1$ and $S^*(b) = b$. Obviously, S is a linear operator. In what follows, we set

$$D_{\tau,1}^{k,\nu}(z) = \frac{1}{b(Q_\tau^{k,\nu})} \int_{Q_\tau^{k,\nu}} b(u) D_k(u,z)\, d\mu(u). \tag{3.3}$$

Then the discrete Riemann sum operator can be re-written into

$$S(f)(x) = \sum_{k=0}^{N} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} \int_{Q_\tau^{k,\nu}} D_k^N(x,y) b(y)\, d\mu(y) D_{\tau,1}^{k,\nu} \mathcal{M}_b(f) \qquad (3.4)$$
$$+ \sum_{k=N+1}^{\infty} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} \int_{Q_\tau^{k,\nu}} D_k^N(x,y) b(y)\, d\mu(y) D_k \mathcal{M}_b(f)(y_\tau^{k,\nu}).$$

We first verify that S is well defined and bounded on $L^2(X)$ via the Littlewood-Paley theorem for the g-function, Lemma 2.3. To do so, let us first establish the following estimate by using Lemma 2.3.

Lemma 3.1. *Let b be a special para-accretive function. Then there exists a constant $C > 0$ such that for all $N \in \mathbb{N}$, all $y_\tau^{k,\nu} \in Q_\tau^{k,\nu}$ and all $f \in L^2(X)$,*

$$\sum_{k=0}^{N} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} \mu(Q_\tau^{k,\nu}) |D_{\tau,1}^{k,\nu} \mathcal{M}_b(f)|^2$$
$$+ \sum_{k=N+1}^{\infty} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} \mu(Q_\tau^{k,\nu}) |D_k \mathcal{M}_b(f)(y_\tau^{k,\nu})|^2 \leqslant C \|f\|_{L^2(X)}^2.$$

Proof. By Lemma 2.3 there exists a family of linear operators $\{\widetilde{D}_k\}_{k=0}^{\infty}$ whose kernels satisfy (i) and (ii) of Definition 2.6 with ϵ replaced by any $\epsilon' \in (0,\epsilon)$ and (2.6) such that for all $f \in L^2(X)$,

$$f = \sum_{l=0}^{\infty} \widetilde{D}_l \mathcal{M}_b D_l \mathcal{M}_b(f).$$

By Lemma 3.1 in [44] (see also [14]), we have that for any $\epsilon'' \in (0,\epsilon')$, there exists a constant $C > 0$ such that

$$|D_k \mathcal{M}_b \widetilde{D}_l(z,x)| \leqslant C 2^{-|k-l|\epsilon''} \frac{2^{-(k\wedge l)\epsilon'}}{(2^{-(k\wedge l)} + \rho(z,x))^{d+\epsilon'}}$$

for all $x,\ z \in X$ and all $k,\ l \in \mathbb{Z}_+$. Note that for all $x \in X$ and any $z,\ y \in Q_\tau^{k,\nu}$, by Lemma 2.1 (iv), we have that $\rho(x,y) + 2^{-(k\wedge l)} \sim 2^{-(k\wedge l)} + \rho(x,z)$, where $j \in \mathbb{N}$ is sufficiently large. Thus, for all $x \in X$, any $y,\ z \in Q_\tau^{k,\nu}$ and all $k,\ l \in \mathbb{N} \cup \{0\}$,

$$|D_k \mathcal{M}_b \widetilde{D}_l(z,x)| \leqslant C 2^{-|k-l|\epsilon''} \frac{2^{-(k\wedge l)\epsilon'}}{(2^{-(k\wedge l)} + \rho(x,y))^{d+\epsilon'}}.$$

From this, $b \in L^\infty(X)$ and $b \in SPF(X)$, it follows that for $k = 0, 1, \cdots, N$,

$$
\begin{aligned}
&\left|D_{\tau,1}^{k,\nu}\mathcal{M}_b(f)\right| \\
&= \left|\int_X \left[\frac{1}{b(Q_\tau^{k,\nu})}\int_{Q_\tau^{k,\nu}} b(z)D_k(z,y)\,d\mu(z)\right] b(y)f(y)\,d\mu(y)\right| \\
&\leqslant C\sum_{l=0}^{\infty}\frac{1}{\mu(Q_\tau^{k,\nu})}\left|\int_{Q_\tau^{k,\nu}}\int_X b(z)(D_k\mathcal{M}_b\widetilde{D}_l)(z,x)\mathcal{M}_bD_l\mathcal{M}_b(f)(x)\,d\mu(x)\,d\mu(z)\right| \\
&\leqslant C\sum_{l=0}^{\infty}2^{-|k-l|\epsilon''}M(D_l\mathcal{M}_b(f))(y)\chi_{Q_\tau^{k,\nu}}(y)
\end{aligned}
\tag{3.5}
$$

and

$$
\begin{aligned}
|D_k\mathcal{M}_b(f)(y_\tau^{k,\nu})| &\leqslant \sum_{l=0}^{\infty}\int_X |(D_k\mathcal{M}_b\widetilde{D}_l)(y_\tau^{k,\nu},x)||\mathcal{M}_bD_l\mathcal{M}_b(f)(x)|\,d\mu(x) \\
&\leqslant C\sum_{l=0}^{\infty}2^{-|k-l|\epsilon''}M(D_l\mathcal{M}_b(f))(y)\chi_{Q_\tau^{k,\nu}}(y),
\end{aligned}
\tag{3.6}
$$

where M is the Hardy-Littlewood maximal function on X. By (3.5), (3.6), the construction of $Q_\tau^{k,\nu}$ (see Lemma 2.1) and Lemma 2.8, we obtain

$$
\begin{aligned}
&\sum_{k=0}^{N}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\mu(Q_\tau^{k,\nu})|D_{\tau,1}^{k,\nu}\mathcal{M}_b(f)|^2 + \sum_{k=N+1}^{\infty}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\mu(Q_\tau^{k,\nu})|D_k\mathcal{M}_b(f)(y_\tau^{k,\nu})|^2 \\
&\leqslant C\sum_{k=0}^{\infty}\int_X\left[\sum_{l=0}^{\infty}2^{-|k-l|\epsilon''}M(D_l\mathcal{M}_b(f))(y)\right]^2 d\mu(y) \\
&\leqslant C\sum_{l=0}^{\infty}\|M(D_l\mathcal{M}_b(f))\|_{L^2(X)}^2 \\
&\leqslant C\sum_{l=0}^{\infty}\|D_l\mathcal{M}_b(f)\|_{L^2(X)}^2 \\
&\leqslant C\|f\|_{L^2(X)}^2,
\end{aligned}
$$

which proves Lemma 3.1. ∎

The next lemma can be proved by a way similar to the proof of Theorem (1.14) in ([11], page 12). The main idea is to combine Lemma 2.4, Lemma 2.1 and Hölder's inequality with a dual argument. We omit the details here; see also [22].

Lemma 3.2. *Suppose that a sequence,* $\{a_\tau^{k,\nu}\}$, *of numbers satisfies*

$$\sum_{k=0}^{\infty}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}|a_\tau^{k,\nu}|^2<\infty.$$

Then the function defined by

$$f(x) = \sum_{\tau\in I_0}\sum_{\nu=1}^{N(0,\tau)} [\mu(Q_\tau^{0,\nu})]^{1/2} a_\tau^{0,\nu} \int_{Q_\tau^{k,\nu}} D_k^N(x,y)b(y)\,d\mu(y)$$
$$+\sum_{k=1}^{\infty}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)} [\mu(Q_\tau^{k,\nu})]^{1/2} a_\tau^{k,\nu} \int_{Q_\tau^{k,\nu}} D_k^N(x,y)b(y)\,d\mu(y)$$

is in $L^2(X)$. *Moreover,*

$$\|f\|_{L^2(X)}^2 \leqslant C\sum_{k=0}^{\infty}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)} |a_\tau^{k,\nu}|^2.$$

Lemma 3.1 and Lemma 3.2 yield the boundedness of the discrete Riemann sum operator S in $L^2(X)$.

Theorem 3.1. *Let* b *be a special para-accretive function. The discrete Riemann sum operator* S *from (3.4) is bounded on* $L^2(X)$. *That is, there is a constant* $C>0$ *such that for all* $f\in L^2(X)$,

$$\|S(f)\|_{L^2(X)} \leqslant C\|f\|_{L^2(X)}.$$

Next we prove that the discrete Riemann sum operator S is invertible and S^{-1} maps $\mathcal{G}(x_0,r,\beta,\gamma)$ into itself. To do this, we define $R=I-S$ and first prove the following theorem.

Theorem 3.2. *Let* b *be a special para-accretive function,* S *be as in (3.4) and* $R=I-S$. *Then* R *is a Calderón-Zygmund singular integral operator,* $R(1)=0=R^*(b)$. *Moreover, its kernel,* $R(x,y)$, *satisfies the following estimates: for any* $\epsilon'\in(0,\epsilon)$ *and some* $\delta>0$, *there exist constants* $C>0$ *and* $C_N>0$ *(both depending on* ϵ'*) such that*

$$|R(x,y)| \leqslant (C2^{-\delta N}+C_N2^{-j\delta})\rho(x,y)^{-d}; \tag{3.7}$$

$$|R(x,y)b^{-1}(y)-R(x,y')b^{-1}(y)| \leqslant (C2^{-\delta N}+C_N2^{-j\delta})\rho(y,y')^{\epsilon'}\rho(x,y)^{-(d+\epsilon')} \tag{3.8}$$

for $\rho(y,y')\leqslant\rho(x,y)/(4A^2)$;

$$|R(x,y)-R(x',y)| \leqslant (C2^{-\delta N}+C_N2^{-j\delta})\rho(x,x')^{\epsilon'}\rho(x,y)^{-(d+\epsilon')} \tag{3.9}$$

for $\rho(x,x')\leqslant\rho(x,y)/(4A^2)$;

$$\begin{aligned}&|[R(x,y)-R(x',y)]b^{-1}(y)-[R(x,y')-R(x',y')]b^{-1}(y')|\\&\quad\leqslant (C2^{-\delta N}+C_N2^{-j\delta})\rho(x,x')^{\epsilon'}\rho(y,y')^{\epsilon'}\rho(x,y)^{-(d+2\epsilon')}\end{aligned} \tag{3.10}$$

for $\rho(x,x')\leqslant\rho(x,y)/(4A^2)$ *and* $\rho(y,y')\leqslant\rho(x,y)/(4A^2)$;

$$|\langle R,f\rangle| \leqslant (C2^{-\delta N}+C_N2^{-j\delta})r^d \tag{3.11}$$

for all $r>0$ *and all continuous function* f *on* $X\times X$ *with* $\operatorname{supp} f\subseteq B(x_1,r)\times B(y_1,r)$, *where* x_1, $y_1\in X$, $\|f\|_{L^\infty(X\times X)}\leqslant 1$, $\|f(\cdot,y)\|_{C_0^\eta(X)}\leqslant r^{-\eta}$ *for all* $y\in X$ *and* $\|f(x,\cdot)\|_{C_0^\eta(X)}\leqslant r^{-\eta}$ *for all* $x\in X$.

Proof. Let $\{S_k\}_{k=0}^{\infty}$ be an approximation to the identity as in Definition 2.6. We let $D_k = 0$ when $k \in \{-1, -2, \cdots\}$, $D_0 = S_0$ and $D_k = S_k - S_{k-1}$ for $k \in \mathbb{N}$. For any given large $N \in \mathbb{N}$ and any $k \in \mathbb{Z}$, let D_k^N be as in (3.2). It is easy to see that

$$I = \sum_{k=0}^{\infty} D_k \mathcal{M}_b$$

in $L^2(X)$. By Coifman's idea, we rewrite this into

$$\begin{aligned} I &= \left(\sum_{k=0}^{\infty} D_k \mathcal{M}_b\right)\left(\sum_{j=0}^{\infty} D_j \mathcal{M}_b\right) \\ &= \sum_{|l|>N} \sum_{k=0}^{\infty} D_{k+l} \mathcal{M}_b D_k \mathcal{M}_b + \sum_{k=0}^{\infty} D_k^N \mathcal{M}_b D_k \mathcal{M}_b. \end{aligned}$$

From this, it follows that

$$\begin{aligned} &R(f)(x) \\ &= \sum_{k=0}^{N} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} \int_{Q_\tau^{k,\nu}} D_k^N(x,y) \left[\mathcal{M}_b D_k \mathcal{M}_b(f)(y) - D_{\tau,1}^{k,\nu} \mathcal{M}_b(f)\right] d\mu(y) \\ &\quad + \sum_{k=N+1}^{\infty} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} \int_{Q_\tau^{k,\nu}} D_k^N(x,y) b(y) \left[D_k \mathcal{M}_b(f)(y) - D_k \mathcal{M}_b(f)(y_\tau^{k,\nu})\right] d\mu(y) \\ &\quad + \sum_{k=0}^{\infty} \sum_{|l|>N} D_{k+l} \mathcal{M}_b D_k \mathcal{M}_b(f)(x) \\ &= G_1(f)(x) + G_2(f)(x) + R_N(f)(x). \end{aligned}$$

For $i = 1, 2$, we denote by $G_i(x, z)$ the kernel of G_i respectively. The estimate for R_N is established in Lemma 3.2 in [44]; see also [20].

We now estimate G_i for $i = 1, 2$ and first estimate G_2. Clearly,

$$G_2(x,z) = \sum_{k=N+1}^{\infty} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} \int_{Q_\tau^{k,\nu}} D_k^N(x,y) b(y) \left[D_k(y,z) - D_k(y_\tau^{k,\nu}, z)\right] b(z)\, d\mu(y).$$

From $b \in L^\infty(X)$,

$$\rho(y, y_\tau^{k,\nu}) \sim 2^{-j-k} \tag{3.12}$$

and the regularity of D_k^N, it follows that

$$\begin{aligned} &|G_2(x,z)| \qquad (3.13) \\ &\leqslant C_N 2^{-j\epsilon} \sum_{k=N+1}^{\infty} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} \int_{Q_\tau^{k,\nu}} \frac{2^{-k\epsilon}}{(2^{-k} + \rho(x,y))^{d+\epsilon}} \frac{2^{-k\epsilon}}{(2^{-k} + \rho(y,z))^{d+\epsilon}}\, d\mu(y) \end{aligned}$$

$$
\begin{aligned}
&\leqslant C_N 2^{-j\epsilon} \sum_{k=N+1}^{\infty} \int_X \frac{2^{-k\epsilon}}{(2^{-k}+\rho(x,y))^{d+\epsilon}} \frac{2^{-k\epsilon}}{(2^{-k}+\rho(y,z))^{d+\epsilon}}\, d\mu(y) \\
&= C_N 2^{-j\epsilon} \sum_{k=N+1}^{\infty} \left\{ \int_{\rho(x,y)\geqslant \frac{\rho(x,z)}{2A}} \frac{2^{-k\epsilon}}{(2^{-k}+\rho(x,y))^{d+\epsilon}} \right. \\
&\quad \left. \times \frac{2^{-k\epsilon}}{(2^{-k}+\rho(y,z))^{d+\epsilon}}\, d\mu(y) + \int_{\rho(y,z)\geqslant \frac{\rho(x,z)}{2A}} \cdots \right\} \\
&\leqslant C_N 2^{-j\epsilon} \sum_{k=N+1}^{\infty} \frac{2^{-k\epsilon}}{(2^{-k}+\rho(x,z))^{d+\epsilon}} \\
&\leqslant C_N 2^{-j\epsilon} \left\{ \rho(x,z)^{-(d+\epsilon)} \sum_{\{k\in\mathbb{Z}^+ : 2^{-k}\leqslant\rho(x,z)\}} 2^{-k\epsilon} + \sum_{\{k\in\mathbb{Z}^+ : 2^{-k}>\rho(x,z)\}} 2^{kd} \right\} \\
&\leqslant C_N 2^{-j\epsilon} \rho(x,z)^{-d},
\end{aligned}
$$

where if $\rho(x,z) > 1$, then the second term is empty. This verifies that $G_2(x,z)$ satisfies (3.7) with the constant $C_N 2^{-j\epsilon}$.

We now write

$$
\begin{aligned}
&G_2(x,z)b^{-1}(z) - G_2(x,z')b^{-1}(z') \\
&\quad = \sum_{k=N+1}^{\infty} \sum_{\tau\in I_k} \sum_{\nu=1}^{N(k,\tau)} \int_{Q_\tau^{k,\nu}} D_k^N(x,y)b(y) \left\{ \left[D_k(y,z) - D_k(y_\tau^{k,\nu},z)\right] \right. \\
&\qquad \left. - \left[D_k(y,z') - D_k(y_\tau^{k,\nu},z')\right] \right\} d\mu(y).
\end{aligned}
$$

We verify that G_2 satisfies (3.8) by considering two cases.

Case 1. $\rho(z,z') \leqslant \frac{1}{2A}(2^{-k}+\rho(y,z))$. In this case, $b \in L^\infty(X)$, (iv) of Definition 2.6 and (3.12) yield that

$$
\begin{aligned}
&|G_2(x,z)b^{-1}(z) - G_2(x,z')b^{-1}(z')| \qquad (3.14) \\
&\leqslant C \sum_{k=N+1}^{\infty} \sum_{\tau\in I_k} \sum_{\nu=1}^{N(k,\tau)} \int_{Q_\tau^{k,\nu}} |D_k^N(x,y)| \left| \left[D_k(y,z) - D_k(y_\tau^{k,\nu},z)\right] \right. \\
&\quad \left. - \left[D_k(y,z') - D_k(y_\tau^{k,\nu},z')\right] \right| d\mu(y) \\
&\leqslant C 2^{-j\epsilon} \sum_{k=N+1}^{\infty} \int_X |D_k^N(x,y)| \left(\frac{\rho(z,z')}{2^{-k}+\rho(y,z)} \right)^{\lambda\epsilon} \frac{2^{-k\epsilon}}{(2^{-k}+\rho(y,z))^{d+\epsilon}}\, d\mu(y) \\
&\leqslant C 2^{-j\epsilon} \sum_{k=N+1}^{\infty} \left\{ \int_{\rho(y,z)\geqslant \frac{1}{2A}\rho(x,z)} \cdots + \int_{\rho(y,z)<\frac{1}{2A}\rho(x,z)} \cdots \right\} \\
&\leqslant C 2^{-j\epsilon} \rho(z,z')^{\lambda\epsilon} \sum_{k=N+1}^{\infty} \frac{2^{-k(1-\lambda)\epsilon}}{(2^{-k}+\rho(x,z))^{d+\epsilon}}
\end{aligned}
$$

$$\leqslant C2^{-j\epsilon}\rho(z,z')^{\lambda\epsilon}\left\{\sum_{\{k\in\mathbb{Z}_+:\,2^{-k}>\rho(x,z)\}} 2^{k(d+\lambda\epsilon)} + \frac{1}{\rho(x,z)^{d+\epsilon}}\sum_{\{k\in\mathbb{Z}_+:\,2^{-k}\leqslant\rho(x,z)\}} 2^{-k(1-\lambda)\epsilon}\right\}$$
$$\leqslant C_N 2^{-j\epsilon}\rho(z,z')^{\lambda\epsilon}\rho(x,z)^{-(d+\lambda\epsilon)},$$

where λ can be any number in $(0,1)$ and if $\rho(x,z)>1$, then the first term is empty. This is a desired estimate.

Case 2. $\frac{1}{2A}(2^{-k}+\rho(y,z))<\rho(z,z')\leqslant\frac{\rho(x,z)}{4A^2}$. In this case, by $b\in L^\infty(X)$, (ii) of Definition 2.6 and (3.12), we have

$$\begin{aligned}
&|\,G_2(x,z)b^{-1}(z)-G_2(x,z')b^{-1}(z')| \qquad (3.15)\\
&\leqslant C2^{-j\epsilon}\sum_{k=N+1}^{\infty}\int_X |D_k^N(x,y)|\left[\frac{2^{-k\epsilon}}{(2^{-k}+\rho(y,z))^{d+\epsilon}}+\frac{2^{-k\epsilon}}{(2^{-k}+\rho(y,z'))^{d+\epsilon}}\right]d\mu(y)\\
&\leqslant C2^{-j\epsilon}\rho(z,z')^{\lambda\epsilon}\sum_{k=N+1}^{\infty}\frac{2^{-k(1-\lambda)\epsilon}}{(2^{-k}+\rho(x,z))^{d+\epsilon}}\\
&\quad\times\int_X\left[\frac{2^{-k\epsilon}}{(2^{-k}+\rho(y,z))^{d+\epsilon}}+\frac{2^{-k\epsilon}}{(2^{-k}+\rho(y,z'))^{d+\epsilon}}\right]d\mu(y)\\
&\leqslant C2^{-j\epsilon}\rho(z,z')^{\lambda\epsilon}\sum_{k=N+1}^{\infty}\frac{2^{-k(1-\lambda)\epsilon}}{(2^{-k}+\rho(x,z))^{d+\epsilon}}\\
&\leqslant C_N 2^{-j\epsilon}\rho(z,z')^{\lambda\epsilon}\rho(x,z)^{-(d+\lambda\epsilon)},
\end{aligned}$$

where λ can be any number in $(0,1)$ and we omit some computation similar to (3.14), which verifies that G_2 satisfies (3.8).

Note that

$$\begin{aligned}
&G_2(x,z)-G_2(x',z)\\
&=\sum_{k=N+1}^{\infty}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\int_{Q_\tau^{k,\nu}}\left[D_k^N(x,y)-D_k^N(x',y)\right]b(y)\\
&\quad\times\left[D_k(y,z)-D_k(y_\tau^{k,\nu},z)\right]b(z)\,d\mu(y).
\end{aligned}$$

To verify G_2 satisfies (3.9), we also consider two cases.

Case 1. $\rho(x,x')\leqslant\frac{1}{2A}(2^{-k}+\rho(x,y))$. In this case, $b\in L^\infty(X)$, (3.12) and (ii) of Definition 2.6 lead us that

$$\begin{aligned}
&|\,G_2(x,z)-G_2(x',z)| \qquad (3.16)\\
&\leqslant C_N 2^{-j\epsilon}\sum_{k=N+1}^{\infty}\int_X\left(\frac{\rho(x,x')}{2^{-k}+\rho(x,y)}\right)^{\lambda\epsilon}\frac{2^{-k\epsilon}}{(2^{-k}+\rho(x,y))^{d+\epsilon}}
\end{aligned}$$

$$\times \frac{2^{-k\epsilon}}{(2^{-k}+\rho(y,z))^{d+\epsilon}}\, d\mu(y)$$
$$\leqslant C_N 2^{-j\epsilon}\rho(x,x')^{\lambda\epsilon}\rho(x,z)^{-(d+\lambda\epsilon)},$$

where λ can be any number in $(0,1)$ and we omit some computation similar to (3.14), which is a desired estimate.

Case 2. $\frac{1}{2A}(2^{-k}+\rho(x,y)) < \rho(x,x') \leqslant \frac{\rho(x,z)}{4A^2}$. In this case, $b \in L^\infty(X)$, (3.12) and (ii) of Definition 2.6 again tell us that

$$\begin{aligned} &|\, G_2(x,z) - G_2(x',z)| \qquad (3.17)\\ &\leqslant C_N 2^{-j\epsilon}\sum_{k=N+1}^{\infty}\int_X \left[|D_k^N(x,y)| + |D_k^N(x',y)|\right]\\ &\quad \times \frac{2^{-k\epsilon}}{(2^{-k}+\rho(y,z))^{d+\epsilon}}\, d\mu(y)\\ &\leqslant C_N 2^{-j\epsilon}\rho(x,x')^{\lambda\epsilon}\sum_{k=N+1}^{\infty}\frac{2^{-k(1-\lambda)\epsilon}}{(2^{-k}+\rho(x,z))^{d+\epsilon}}\\ &\quad \times \int_X \left[|D_k^N(x,y)| + |D_k^N(x',y)|\right] d\mu(y)\\ &\leqslant C_N 2^{-j\epsilon}\rho(x,x')^{\lambda\epsilon}\sum_{k=N+1}^{\infty}\frac{2^{-k(1-\lambda)\epsilon}}{(2^{-k}+\rho(x,z))^{d+\epsilon}}\\ &\leqslant C_N 2^{-j\epsilon}\rho(x,x')^{\lambda\epsilon}\rho(x,z)^{-(d+\lambda\epsilon)}, \end{aligned}$$

where λ can be any number in $(0,1)$ and some computation similar to (3.14) is omitted, which verifies G_2 satisfies (3.9).

We now verify that $G_2(x,z)$ satisfies (3.10) when $\rho(x,x') \leqslant \frac{\rho(x,z)}{4A^2}$ and $\rho(z,z') \leqslant \frac{\rho(x,z)}{4A^2}$. To this end, we write

$$\begin{aligned} &[G_2(x,z) - G_2(x',z)]b^{-1}(z) - [G_2(x,z') - G_2(x',z')]b^{-1}(z')\\ &= \sum_{k=N+1}^{\infty}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\int_{Q_\tau^{k,\nu}} \left[D_k^N(x,y) - D_k^N(x',y)\right] b(y)\left\{\left[D_k(y,z) - D_k(y_\tau^{k,\nu},z)\right]\right.\\ &\quad \left. - \left[D_k(y,z') - D_k(y_\tau^{k,\nu},z')\right]\right\} d\mu(y). \end{aligned}$$

Now, if j is large enough, then $\rho(x,x') \geqslant \frac{1}{2A}(2^{-k}+\rho(x,y))$ and

$$\rho(z,z') \geqslant \frac{1}{2A}(2^{-k}+\rho(y,z))$$

imply

$$\begin{aligned} \rho(x,x') + \rho(z,z') &\geqslant \frac{1}{2A}\left[2^{1-k} + \rho(x,y) + \rho(y,z)\right]\\ &\geqslant \frac{1}{2A^2}\rho(x,z), \end{aligned}$$

which contradicts to $\rho(x,x')\leqslant\frac{\rho(x,z)}{4A^2}$ and $\rho(z,z')\leqslant\frac{\rho(x,z)}{4A^2}$. Thus, we still have three cases:

(i) $\rho(x,x')\leqslant\frac{1}{2A}(2^{-k}+\rho(x,y))$ and $\rho(z,z')\leqslant\frac{1}{2A}(2^{-k}+\rho(y,z))$;

(ii) $\rho(x,x')\leqslant\frac{1}{2A}(2^{-k}+\rho(x,y))$ and $\rho(z,z')>\frac{1}{2A}(2^{-k}+\rho(y,z))$;

(iii) $\rho(x,x')>\frac{1}{2A}(2^{-k}+\rho(x,y))$ and $\rho(z,z')\leqslant\frac{1}{2A}(2^{-k}+\rho(y,z))$.

For the case (i), by (3.12) and Definition 2.6 (ii) and (iv), we obtain that for any $\lambda\in(0,1)$,

$$\begin{aligned}&\left|\,D_k^N(x,y)-D_k^N(x',y)\right|\left|[D_k(y,z)-D_k(y_\tau^{k,\nu},z)]-[D_k(y,z')-D_k(y_\tau^{k,\nu},z')]\right|\\&\leqslant C_N2^{-j\epsilon}\left(\frac{\rho(x,x')}{2^{-k}+\rho(x,y)}\right)^{\epsilon}\frac{2^{-k\epsilon}}{(2^{-k}+\rho(x,y))^{d+\epsilon}}\\&\quad\times\left(\frac{\rho(z,z')}{2^{-k}+\rho(y,z)}\right)^{\lambda\epsilon}\frac{2^{-k\epsilon}}{(2^{-k}+\rho(y,z))^{d+\epsilon}}.\end{aligned}$$

For the case (ii), noting that $\frac{\rho(x,z)}{4A^2}\geqslant\rho(z,z')>\frac{1}{2A}(2^{-k}+\rho(y,z))$ implies that $\rho(z,z')>2^{-k-1}A^{-1}$ and $\rho(x,y)>\frac{\rho(x,z)}{2A}$, we have

$$\begin{aligned}&\left|\,D_k^N(x,y)-D_k^N(x',y)\right|\left|[D_k(y,z)-D_k(y_\tau^{k,\nu},z)]-[D_k(y,z')-D_k(y_\tau^{k,\nu},z')]\right|\\&\leqslant C_N2^{-j\epsilon}\left(\frac{\rho(x,x')}{2^{-k}+\rho(x,y)}\right)^{\epsilon}\frac{2^{-k\epsilon}}{(2^{-k}+\rho(x,y))^{d+\epsilon}}\\&\quad\times\left[\frac{2^{-k\epsilon}}{(2^{-k}+\rho(y,z))^{d+\epsilon}}+\frac{2^{-k\epsilon}}{(2^{-k}+\rho(y,z'))^{d+\epsilon}}\right]\\&\leqslant C_N2^{-j\epsilon}\left(\frac{\rho(x,x')}{2^{-k}+\rho(x,z)}\right)^{\epsilon}\left(\frac{\rho(z,z')}{2^{-k}+\rho(x,z)}\right)^{\lambda\epsilon}\frac{2^{-k(1-\lambda)\epsilon}}{(2^{-k}+\rho(x,z))^{d+(1-\lambda)\epsilon}}\\&\quad\times\left[\frac{2^{-k\epsilon}}{(2^{-k}+\rho(y,z))^{d+\epsilon}}+\frac{2^{-k\epsilon}}{(2^{-k}+\rho(y,z'))^{d+\epsilon}}\right],\end{aligned}$$

where λ can be any positive number in $(0,1)$. For the last case (iii), noting that $\frac{\rho(x,z)}{4A^2}\geqslant\rho(x,x')>\frac{1}{2A}(2^{-k}+\rho(x,y))$ implies that $\rho(x,x')>2^{-k-1}A^{-1}$ and $\rho(y,z)>\frac{\rho(x,z)}{2A}$, we obtain that for any $\lambda\in(0,1)$,

$$\begin{aligned}&\left|\,D_k^N(x,y)-D_k^N(x',y)\right|\left|[D_k(y,z)-D_k(y_\tau^{k,\nu},z)]-[D_k(y,z')-D_k(y_\tau^{k,\nu},z')]\right|\\&\leqslant C_N2^{-j\epsilon}\left(\frac{\rho(z,z')}{2^{-k}+\rho(x,z)}\right)^{\epsilon}\frac{2^{-k\epsilon}}{(2^{-k}+\rho(x,z))^{d+\epsilon}}\\&\quad\times\left[\frac{2^{-k\epsilon}}{(2^{-k}+\rho(x,y))^{d+\epsilon}}+\frac{2^{-k\epsilon}}{(2^{-k}+\rho(x',y))^{d+\epsilon}}\right]\\&\leqslant C_N2^{-j\epsilon}\left(\frac{\rho(x,x')}{2^{-k}+\rho(x,z)}\right)^{\lambda\epsilon}\left(\frac{\rho(z,z')}{2^{-k}+\rho(x,z)}\right)^{\epsilon}\frac{2^{-k(1-\lambda)\epsilon}}{(2^{-k}+\rho(x,z))^{d+(1-\lambda)\epsilon}}\\&\quad\times\left[\frac{2^{-k\epsilon}}{(2^{-k}+\rho(x,y))^{d+\epsilon}}+\frac{2^{-k\epsilon}}{(2^{-k}+\rho(x',y))^{d+\epsilon}}\right].\end{aligned}$$

Combining these estimates with $b \in L^\infty(X)$ yields that

$$
\begin{aligned}
&| \,[G_2(x,z) - G_2(x',z)]b^{-1}(z) - [G_2(x,z') - G_2(x',z')]b^{-1}(z')| \qquad (3.18)\\
&\leqslant C \sum_{k=N+1}^{\infty} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} \int_{Q_\tau^{k,\nu}} \big|[D_k^N(x,y) - D_k^N(x',y)]\,\big\{[D_k(y,z) - D_k(y_\tau^{k,\nu},z)]\\
&\quad - [D_k(y,z') - D_k(y_\tau^{k,\nu},z')]\big\}\big|\, d\mu(y)\\
&\leqslant C_N 2^{-j\epsilon} \sum_{k=N+1}^{\infty} \left\{ \int_X \left(\frac{\rho(x,x')}{2^{-k}+\rho(x,y)}\right)^{\lambda\epsilon} \frac{2^{-k\epsilon}}{(2^{-k}+\rho(x,y))^{d+\epsilon}}\right.\\
&\quad \times \left(\frac{\rho(z,z')}{2^{-k}+\rho(y,z)}\right)^{\lambda\epsilon} \frac{2^{-k\epsilon}}{(2^{-k}+\rho(y,z))^{d+\epsilon}}\, d\mu(y)\\
&\quad + \int_X \left(\frac{\rho(x,x')}{2^{-k}+\rho(x,z)}\right)^{\lambda\epsilon} \left(\frac{\rho(z,z')}{2^{-k}+\rho(x,z)}\right)^{\lambda\epsilon} \frac{2^{-k(1-\lambda)\epsilon}}{(2^{-k}+\rho(x,z))^{d+(1-\lambda)\epsilon}}\\
&\quad \times \left[\frac{2^{-k\epsilon}}{(2^{-k}+\rho(y,z))^{d+\epsilon}} + \frac{2^{-k\epsilon}}{(2^{-k}+\rho(y,z'))^{d+\epsilon}}\right] d\mu(y)\\
&\quad + \int_X \left(\frac{\rho(x,x')}{2^{-k}+\rho(x,z)}\right)^{\lambda\epsilon} \left(\frac{\rho(z,z')}{2^{-k}+\rho(x,z)}\right)^{\lambda\epsilon} \frac{2^{-k(1-\lambda)\epsilon}}{(2^{-k}+\rho(x,z))^{d+(1-\lambda)\epsilon}}\\
&\quad \left.\times \left[\frac{2^{-k\epsilon}}{(2^{-k}+\rho(x,y))^{d+\epsilon}} + \frac{2^{-k\epsilon}}{(2^{-k}+\rho(x',y))^{d+\epsilon}}\right] d\mu(y)\right\}\\
&\leqslant C_N 2^{-j\epsilon} \rho(x,x')^{\lambda\epsilon} \rho(z,z')^{\lambda\epsilon} \sum_{k=N+1}^{\infty} \frac{2^{-k(1-\lambda)\epsilon}}{(2^{-k}+\rho(x,z))^{d+(1+\lambda)\epsilon}}\\
&\quad \times \left\{ \int_{\rho(x,y)\geqslant \rho(x,z)/(2A)} \frac{2^{-k\epsilon}}{(2^{-k}+\rho(y,z))^{d+\epsilon}}\, d\mu(y)\right.\\
&\quad \left. + \int_{\rho(y,z)\geqslant \rho(x,z)/(2A)} \frac{2^{-k\epsilon}}{(2^{-k}+\rho(x,y))^{d+\epsilon}}\, d\mu(y) + 1\right\}\\
&\leqslant C_N 2^{-j\epsilon} \rho(x,x')^{\lambda\epsilon} \rho(z,z')^{\lambda\epsilon} \sum_{k=N+1}^{\infty} \frac{2^{-k(1-\lambda)\epsilon}}{(2^{-k}+\rho(x,z))^{d+(1+\lambda)\epsilon}}\\
&\leqslant C_N 2^{-j\epsilon} \rho(x,x')^{\lambda\epsilon} \rho(z,z')^{\lambda\epsilon} \rho(x,z)^{-(d+2\lambda\epsilon)},
\end{aligned}
$$

where λ can be any positive number in $(0,1)$ and we omit some computation similar to (3.14). This verifies that the kernel of G_2 satisfies (3.10) with the constant $C_N 2^{-j\epsilon}$.

Now, we prove that G_2 satisfies (3.11). Suppose that f is a continuous function on $X \times X$ with $\operatorname{supp} f \subset B(x_1,r) \times B(z_1,r)$ for some x_1 and $z_1 \in X$, $\|f\|_{L^\infty(X\times X)} \leqslant 1$, $\|f(\cdot,y)\|_{C_0^\eta(X)} \leqslant r^{-\eta}$ and $\|f(x,\cdot)\|_{C_0^\eta(X)} \leqslant r^{-\eta}$ for all x and

$y \in X$. Write

$$\begin{aligned}
&G_2(x,z)\\
&= \sum_{k=N+1}^{\infty} \sum_{\tau\in I_k} \sum_{\nu=1}^{N(k,\tau)} \int_{Q_\tau^{k,\nu}} D_k^N(x,y)b(y)\left[D_k(y,z) - D_k(y_\tau^{k,\nu},z)\right] b(z)\,d\mu(y)\\
&= \sum_{k=N+1}^{\infty} G_2^k(x,z).
\end{aligned}$$

By the proof of (3.13), we have

$$\begin{aligned}
|\langle G_2^k, f\rangle| &= \left|\int_X \int_X G_2^k(x,z) f(x,z)\,d\mu(x)\,d\mu(z)\right| \qquad (3.19)\\
&\leqslant C_N 2^{-j\epsilon} \int_X \int_X \frac{2^{-k\epsilon}}{(2^{-k}+\rho(x,z))^{d+\epsilon}} |f(x,z)|\,d\mu(x)\,d\mu(z)\\
&\leqslant C_N 2^{-j\epsilon} \|f\|_{L^\infty(X\times X)} r^d \leqslant C_N 2^{-j\epsilon} r^d.
\end{aligned}$$

On the other hand, if $k \in \mathbb{N}$, by

$$\int_X D_k(y,z)b(z)\,d\mu(z) = 0$$

for any $y \in X$ and $b \in L^\infty(X)$, we then have

$$\begin{aligned}
&|\langle G_2^k, f\rangle| \qquad (3.20)\\
&= \Bigg|\sum_{\tau\in I_k} \sum_{\nu=1}^{N(k,\tau)} \int_X \int_X \int_{Q_\tau^{k,\nu}} D_k^N(x,y)b(y)\left[D_k(y,z)\right.\\
&\quad \left. - D_k(y_\tau^{k,\nu},z)\right] b(z) f(x,z)\,d\mu(y)\,d\mu(x)\,d\mu(z)\Bigg|\\
&= \Bigg|\sum_{\tau\in I_k} \sum_{\nu=1}^{N(k,\tau)} \int_X \int_X \int_{Q_\tau^{k,\nu}} D_k^N(x,y)b(y)\left[D_k(y,z) - D_k(y_\tau^{k,\nu},z)\right] b(z)\\
&\quad \times [f(x,z) - f(x,y)]\,d\mu(y)\,d\mu(x)\,d\mu(z)\Bigg|\\
&\leqslant C r^{-\eta} 2^{-j\epsilon} \int_{B(x_0,r)} \left\{\int_X |D_k^N(x,y)|\right.\\
&\quad \times \left.\left[\int_X \frac{2^{-2k\epsilon}}{(2^{-k}+\rho(y,z))^{d+2\epsilon}} \rho(y,z)^\eta\,d\mu(z)\right] d\mu(y)\right\} d\mu(x)\\
&\leqslant C 2^{-j\epsilon} 2^{-k\eta} r^{-\eta} r^d,
\end{aligned}$$

where $\eta < 2\epsilon$. We also have that for $k \geqslant N+1$,

$$
\begin{aligned}
&|\langle G_2^k, f\rangle| \qquad (3.21)\\
&= \Bigg|\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\int_X\int_X\int_{Q_\tau^{k,\nu}} D_k^N(x,y)b(y)\left[D_k(y,z)\right.\\
&\quad \left.-D_k(y_\tau^{k,\nu},z)\right] b(z)f(x,z)\,d\mu(y)\,d\mu(x)\,d\mu(z)\Bigg|\\
&\leqslant C_N 2^{-j\epsilon}\int_X\int_X\int_X \frac{2^{-2k\epsilon}}{(2^{-k}+\rho(y,z))^{d+2\epsilon}}|D_k^N(x,y)f(x,z)|\,d\mu(y)\,d\mu(x)\,d\mu(z)\\
&\leqslant C_N 2^{-j\epsilon}2^{kd}\|f\|_{L^\infty(X\times X)}r^{2d}\\
&\leqslant C_N 2^{-j\epsilon}2^{kd}r^{2d}.
\end{aligned}
$$

The geometric means of (3.19) and (3.20), and of (3.19) and (3.21) respectively yield

$$|\langle G_2^k, f\rangle| \leqslant C2^{-j\epsilon}2^{-k\eta'}r^{-\eta'}r^d$$

and

$$|\langle G_2^k, f\rangle| \leqslant C2^{-j\epsilon}2^{k\eta''}r^{\eta''}r^d,$$

where $0<\eta',\ \eta''<\eta<2\epsilon$. From this, it follows that

$$
\begin{aligned}
|\langle G_2, f\rangle| &\leqslant \sum_{k=N+1}^{\infty}|\langle G_2^k, f\rangle| \qquad (3.22)\\
&\leqslant C\sum_{\{k\in\mathbb{N}:\ 2^{-k}>r\}} 2^{-j\epsilon}2^{k\eta''}r^{\eta''}r^d + C\sum_{\{k\in\mathbb{N}:\ 2^{-k}\leqslant r\}} 2^{-j\epsilon}2^{-k\eta'}r^{-\eta'}r^d\\
&\leqslant C_N 2^{-j\epsilon}r^d,
\end{aligned}
$$

where the first term is empty if $r>1$. Thus, G_2 satisfies (3.11) with the constant $C_N 2^{-j\epsilon}$. So far, we have finished the estimates for G_2.

We now begin to estimate G_1. By the similarity, we only give an outline. Obviously, we have

$$
\begin{aligned}
G_1(x,z) &= \sum_{k=0}^{N}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\int_{Q_\tau^{k,\nu}} D_k^N(x,y)b(y) \qquad (3.23)\\
&\quad\times\left[D_k(y,z)-\frac{1}{b(Q_\tau^{k,\nu})}\int_{Q_\tau^{k,\nu}} b(u)D_k(u,z)\,d\mu(u)\right]b(z)\,d\mu(y)\\
&= \sum_{k=0}^{N}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\frac{1}{b(Q_\tau^{k,\nu})}\int_{Q_\tau^{k,\nu}}\int_{Q_\tau^{k,\nu}} D_k^N(x,y)b(y)\\
&\quad\times\left[D_k(y,z)-D_k(u,z)\right]b(u)b(z)\,d\mu(u)\,d\mu(y).
\end{aligned}
$$

Since $y,\ u \in Q_\tau^{k,\nu}$, we then have

$$\rho(y,u) \sim 2^{-j-k}. \tag{3.24}$$

From this, (3.24) and $b \in SPF(X)$, similarly to (3.13), it immediately follows that

$$\begin{aligned} |G_1(x,z)| &\leqslant C2^{-j\epsilon}\sum_{k=0}^{N}\int_X |D_k^N(x,y)| \frac{2^{-k\epsilon}}{(2^{-k}+\rho(y,z))^{d+\epsilon}}\,d\mu(y) \\ &\leqslant C_N 2^{-j\epsilon}\sum_{k=0}^{N}\frac{2^{-k\epsilon}}{(2^{-k}+\rho(x,z))^{d+\epsilon}} \\ &\leqslant C_N 2^{-j\epsilon}\rho(x,z)^{-d}, \end{aligned} \tag{3.25}$$

which shows G_1 satisfies the estimate (3.7).

We now verify G_1 satisfies (3.8). By (3.23), we have

$$\begin{aligned} &G_1(x,z)b^{-1}(z) - G_1(x,z')b^{-1}(z') \\ &= \sum_{k=0}^{N}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\frac{1}{b(Q_\tau^{k,\nu})}\int_{Q_\tau^{k,\nu}}\int_{Q_\tau^{k,\nu}} D_k^N(x,y)b(y) \\ &\quad\times\{[D_k(y,z)-D_k(u,z)]-[D_k(y,z')-D_k(u,z')]\}\,b(u)\,d\mu(u)\,d\mu(y). \end{aligned}$$

Similarly to the case for G_2, we consider two cases.

Case 1. $\rho(z,z') \leqslant \frac{1}{2A}(2^{-k}+\rho(y,z))$. In this case, similarly to the estimate of (3.14), (3.24) and $b \in SPF(X)$ yield that

$$\begin{aligned} &|\ G_1(x,z)b^{-1}(z) - G_1(x,z')b^{-1}(z')| \\ &\leqslant C2^{-j\epsilon}\sum_{k=0}^{N}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\frac{1}{\mu(Q_\tau^{k,\nu})}\int_{Q_\tau^{k,\nu}}\int_{Q_\tau^{k,\nu}}|D_k^N(x,y)| \\ &\quad\times\left(\frac{\rho(z,z')}{2^{-k}+\rho(y,z)}\right)^{\lambda\epsilon}\frac{2^{-k\epsilon}}{(2^{-k}+\rho(y,z))^{d+\epsilon}}\,d\mu(u)\,d\mu(y) \\ &\leqslant C_N 2^{-j\epsilon}\rho(z,z')^{\lambda\epsilon}\sum_{k=0}^{N}\frac{2^{-k(1-\lambda)\epsilon}}{(2^{-k}+\rho(x,z))^{d+\epsilon}} \\ &\leqslant C_N 2^{-j\epsilon}\rho(z,z')^{\lambda\epsilon}\frac{1}{\rho(x,z)^{d+\lambda\epsilon}}, \end{aligned}$$

where λ can be any positive number in $(0,1)$.

Case 2. $\frac{1}{2A}(2^{-k}+\rho(y,z)) < \rho(z,z') \leqslant \frac{\rho(x,z)}{4A^2}$. In this case, similarly to the estimate of (3.15), (3.24) and $b \in SPF(X)$ yield that

$$
\begin{aligned}
&|\, G_1(x,z)b^{-1}(z) - G_1(x,z')b^{-1}(z')| \\
&\leqslant C2^{-j\epsilon}\sum_{k=0}^{N}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)} \frac{1}{\mu(Q_\tau^{k,\nu})}\int_{Q_\tau^{k,\nu}}\int_{Q_\tau^{k,\nu}} |D_k^N(x,y)| \\
&\quad\times\left[\frac{2^{-k\epsilon}}{(2^{-k}+\rho(y,z))^{d+\epsilon}} + \frac{2^{-k\epsilon}}{(2^{-k}+\rho(y,z'))^{d+\epsilon}}\right] d\mu(u)\, d\mu(y) \\
&\leqslant C_N 2^{-j\epsilon}\rho(z,z')^{\lambda\epsilon}\sum_{k=0}^{N}\frac{2^{-k(1-\lambda)\epsilon}}{(2^{-k}+\rho(x,z))^{d+\epsilon}} \\
&\quad\times\int_X\left[\frac{2^{-k\epsilon}}{(2^{-k}+\rho(y,z))^{d+\epsilon}} + \frac{2^{-k\epsilon}}{(2^{-k}+\rho(y,z'))^{d+\epsilon}}\right] d\mu(y) \\
&\leqslant C_N 2^{-j\epsilon}\rho(z,z')^{\lambda\epsilon}\sum_{k=0}^{N}\frac{2^{-k(1-\lambda)\epsilon}}{(2^{-k}+\rho(x,z))^{d+\epsilon}} \\
&\leqslant C_N 2^{-j\epsilon}\rho(z,z')^{\lambda\epsilon}\frac{1}{\rho(x,z)^{d+\lambda\epsilon}},
\end{aligned}
$$

where λ can be any positive number in $(0,1)$. This verifies that G_1 satisfies (3.8).

Note that

$$
\begin{aligned}
&G_1(x,z) - G_1(x',z) \\
&= \sum_{k=0}^{N}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)} \frac{1}{b(Q_\tau^{k,\nu})}\int_{Q_\tau^{k,\nu}}\int_{Q_\tau^{k,\nu}} [D_k^N(x,y) - D_k^N(x',y)]\, b(y) \\
&\quad\times [D_k(y,z) - D_k(u,z)]\, b(u)b(z)\, d\mu(u)\, d\mu(y).
\end{aligned}
$$

To verify G_1 satisfies (3.9), we also consider two cases.

Case 1. $\rho(x,x') \leqslant \frac{1}{2A}(2^{-k}+\rho(x,y))$. In this case, similarly to (3.16), by (3.24) and $b \in SPF(X)$, we obtain

$$
\begin{aligned}
&|\, G_1(x,z) - G_1(x',z)| \\
&\leqslant C_N 2^{-j\epsilon}\sum_{k=0}^{N}\int_X \left(\frac{\rho(x,x')}{2^{-k}+\rho(x,y)}\right)^{\lambda\epsilon} \frac{2^{-k\epsilon}}{(2^{-k}+\rho(x,y))^{d+\epsilon}} \\
&\quad\times\frac{2^{-k\epsilon}}{(2^{-k}+\rho(y,z))^{d+\epsilon}}\, d\mu(y) \\
&\leqslant C_N 2^{-j\epsilon}\rho(x,x')^{\lambda\epsilon}\sum_{k=0}^{N}\frac{2^{-k(1-\lambda)\epsilon}}{(2^{-k}+\rho(x,z))^{d+\epsilon}} \\
&\leqslant C_N 2^{-j\epsilon}\rho(x,x')^{\lambda\epsilon}\frac{1}{\rho(x,z)^{d+\lambda\epsilon}},
\end{aligned}
$$

where λ can be any positive number in $(0,1)$.

Case 2. $\frac{1}{2A}(2^{-k}+\rho(x,y)) < \rho(x,x') \leqslant \frac{\rho(x,z)}{4A^2}$. In this case, similarly to (3.17), (3.24) and $b \in SPF(X)$ tell us that

$$\begin{aligned}
&| G_1(x,z) - G_1(x',z)| \\
&\leqslant C_N 2^{-j\epsilon} \sum_{k=0}^{N} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} \frac{1}{\mu(Q_\tau^{k,\nu})} \int_{Q_\tau^{k,\nu}} \int_{Q_\tau^{k,\nu}} [|D_k^N(x,y)| + |D_k^N(x',y)|] \\
&\quad \times \frac{2^{-k\epsilon}}{(2^{-k}+\rho(y,z))^{d+\epsilon}} \, d\mu(u) \, d\mu(y) \\
&\leqslant C_N 2^{-j\epsilon} \rho(x,x')^{\lambda\epsilon} \sum_{k=0}^{N} \frac{2^{-k(1-\lambda)\epsilon}}{(2^{-k}+\rho(x,z))^{d+\epsilon}} \\
&\quad \times \int_X [|D_k^N(x,y)| + |D_k^N(x',y)|] \, d\mu(y) \\
&\leqslant C_N 2^{-j\epsilon} \rho(x,x')^{\lambda\epsilon} \sum_{k=0}^{N} \frac{2^{-k(1-\lambda)\epsilon}}{(2^{-k}+\rho(x,z))^{d+\epsilon}} \\
&\leqslant C_N 2^{-j\epsilon} \rho(x,x')^{\lambda\epsilon} \frac{1}{\rho(x,z)^{d+\lambda\epsilon}},
\end{aligned}$$

where λ can be any positive number in $(0,1)$, which verifies that G_1 satisfies (3.9).

We now show that G_1 satisfies (3.10). To this end, we write

$$\begin{aligned}
&[G_1(x,z) - G_1(x',z)] b^{-1}(z) - [G_1(x,z') - G_1(x',z')] b^{-1}(z') \\
&= \sum_{k=0}^{N} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} \frac{1}{b(Q_\tau^{k,\nu})} \int_{Q_\tau^{k,\nu}} \int_{Q_\tau^{k,\nu}} [D_k^N(x,y) - D_k^N(x',y)] \, b(y) \\
&\quad \times \{[D_k(y,z) - D_k(u,z)] - [D_k(y,z') - D_k(u,z')]\} \, b(u) \, d\mu(u) \, d\mu(y).
\end{aligned}$$

Since $\rho(x,x')$, $\rho(z,z') \leqslant \frac{\rho(x,z)}{4A^2}$, similarly to the case for G_2, we also have three cases:

(i) $\rho(x,x') \leqslant \frac{1}{2A}(2^{-k}+\rho(x,y))$ and $\rho(z,z') \leqslant \frac{1}{2A}(2^{-k}+\rho(y,z))$;

(ii) $\rho(x,x') \leqslant \frac{1}{2A}(2^{-k}+\rho(x,y))$ and $\rho(z,z') > \frac{1}{2A}(2^{-k}+\rho(y,z))$;

(iii) $\rho(x,x') > \frac{1}{2A}(2^{-k}+\rho(x,y))$ and $\rho(z,z') \leqslant \frac{1}{2A}(2^{-k}+\rho(y,z))$.

The estimate (3.24), $b \in SPF(X)$ and an argument similar to that for (3.18) yield that

$$[\, G_1(x,z)-G_1(x',z)]\,b^{-1}(z)-[G_1(x,z')-G_1(x',z')]\,b^{-1}(z')$$

$$\leqslant C_N 2^{-j\epsilon}\sum_{k=0}^{N}\left\{\int_X\left(\frac{\rho(x,x')}{2^{-k}+\rho(x,y)}\right)^{\lambda\epsilon}\frac{2^{-k\epsilon}}{(2^{-k}+\rho(x,y))^{d+\epsilon}}\right.$$

$$\times\left(\frac{\rho(z,z')}{2^{-k}+\rho(y,z)}\right)^{\lambda\epsilon}\frac{2^{-k\epsilon}}{(2^{-k}+\rho(y,z))^{d+\epsilon}}\,d\mu(y)$$

$$+\int_X\left(\frac{\rho(x,x')}{2^{-k}+\rho(x,z)}\right)^{\lambda\epsilon}\left(\frac{\rho(z,z')}{2^{-k}+\rho(x,z)}\right)^{\lambda\epsilon}\frac{2^{-k(1-\lambda)\epsilon}}{(2^{-k}+\rho(x,z))^{d+(1-\lambda)\epsilon}}$$

$$\times\left[\frac{2^{-k\epsilon}}{(2^{-k}+\rho(y,z))^{d+\epsilon}}+\frac{2^{-k\epsilon}}{(2^{-k}+\rho(y,z'))^{d+\epsilon}}\right]d\mu(y)$$

$$+\int_X\left(\frac{\rho(x,x')}{2^{-k}+\rho(x,z)}\right)^{\lambda\epsilon}\left(\frac{\rho(z,z')}{2^{-k}+\rho(x,z)}\right)^{\lambda\epsilon}\frac{2^{-k(1-\lambda)\epsilon}}{(2^{-k}+\rho(x,z))^{d+(1-\lambda)\epsilon}}$$

$$\left.\times\left[\frac{2^{-k\epsilon}}{(2^{-k}+\rho(x,y))^{d+\epsilon}}+\frac{2^{-k\epsilon}}{(2^{-k}+\rho(x',y))^{d+\epsilon}}\right]d\mu(y)\right\}$$

$$\leqslant C_N 2^{-j\epsilon}\rho(x,x')^{\lambda\epsilon}\rho(z,z')^{\lambda\epsilon}\sum_{k=0}^{N}\frac{2^{-k(1-\lambda)\epsilon}}{(2^{-k}+\rho(x,z))^{d+(1+\lambda)\epsilon}}$$

$$\leqslant C_N 2^{-j\epsilon}\rho(x,x')^{\lambda\epsilon}\rho(z,z')^{\lambda\epsilon}\rho(x,z)^{-(d+2\lambda\epsilon)},$$

where λ can be any positive number in $(0,1)$, which verifies that G_1 satisfies (3.10).

Finally we show that G_1 satisfies (3.11). We write

$$G_1(x,z)=\sum_{k=0}^{N}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\frac{1}{b(Q_\tau^{k,\nu})}\int_{Q_\tau^{k,\nu}}\int_{Q_\tau^{k,\nu}}D_k^N(x,y)b(y)$$

$$\times\,[D_k(y,z)-D_k(u,z)]\,b(u)b(z)\,d\mu(u)\,d\mu(y)$$

$$=\sum_{k=0}^{N}G_1^k(x,z).$$

Let f be the same as in the theorem. By $b\in SPF(X)$, the estimate (3.24), the proof of (3.25) and an argument similar to that for (3.19), we obtain

$$|\langle G_1^k,f\rangle|\leqslant C_N 2^{-j\epsilon}r^d.$$

From this, it is easy to deduce that G_1 satisfies (3.11). This finishes the proof of Theorem 3.2. ■

Note that $R(1)=0=R^*(b)$ by our special choices and

$$S^{-1}=\sum_{m=0}^{\infty}R^m. \tag{3.26}$$

As a simple corollary of Theorem 3.2, Lemma 2.2 and the Tb-theorem in [6] (see also [4], we have the following conclusion.

Theorem 3.3. *Let b be a special para-accretive function and S be as in (3.4). If j and N are large positive integers, then S has a bounded inverse on any space $\mathcal{G}_0(x_0, r, \beta, \gamma)$ ($x_0 \in X$, $r > 0$, $0 < \beta$, $\gamma < \epsilon$) as well as on each of the spaces $L^p(X)$ with $1 < p < \infty$. In other words, there exist constants $C > 0$ (depending on the space of test functions, but not on f), and constants $C_p > 0$ such that*

$$\|S^{-1}(f)\|_{\mathcal{G}(x_0,d,\beta,\gamma)} \leqslant C\|f\|_{\mathcal{G}(x_0,d,\beta,\gamma)}$$

and

$$\|S^{-1}(f)\|_{L^p(X)} \leqslant C_p\|f\|_{L^p(X)}.$$

Now we can state and prove the main result of this section, that is, the following inhomogeneous discrete Calderón reproducing formulas.

Theorem 3.4. *Let b be a special para-accretive function. Suppose that $\{S_k\}_{k\in\mathbb{Z}_+}$ is an approximation to the identity of order $\epsilon \in (0,\theta]$ as in Definition 2.6 and $\{D_k\}_{k\in\mathbb{Z}_+}$ is as in Section 2. Then there exist a fixed large integer $N \in \mathbb{N}$ (and $j \in \mathbb{N}$) and a family of functions $\widetilde{D}_k(x,y)$ for $k \in \mathbb{Z}_+$ such that for any fixed $y_\tau^{k,\nu} \in Q_\tau^{k,\nu}$ with $k = N+1, \cdots, \tau \in I_k$ and $\nu \in \{1, \cdots, N(k,\tau)\}$ and all $f \in \mathcal{G}(\beta_1, \gamma_1)$ with $0 < \beta_1 < \epsilon$ and $0 < \gamma_1 < \epsilon$,*

$$f(x) = \sum_{k=0}^{N}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)} \int_{Q_\tau^{k,\nu}} \widetilde{D}_k(x,y)b(y)\,d\mu(y) D_{\tau,1}^{k,\nu}\mathcal{M}_b(f) \tag{3.27}$$
$$+ \sum_{k=N+1}^{\infty}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)} \int_{Q_\tau^{k,\nu}} \widetilde{D}_k(x,y)b(y)\,d\mu(y) D_k\mathcal{M}_b(f)(y_\tau^{k,\nu}),$$

where the series converge in the norms of $L^p(X)$, $1 < p < \infty$, and $\mathcal{G}(\beta_1', \gamma_1')$ for $0 < \beta_1' < \beta_1$ and $0 < \gamma_1' < \gamma_1$; $D_{\tau,1}^{k,\nu}$ for $k = 0, 1, \cdots, N$ is a linear operator having the kernel $D_{\tau,1}^{k,\nu}$ defined by (3.3); Moreover, there is a constant $C > 0$ such that the function $\widetilde{D}_k(x,y)$ for $k = 1, \cdots, N$ satisfies

(i) $\left|\widetilde{D}_k(x,y)\right| \leqslant C\frac{1}{(1+\rho(x,y))^{d+\epsilon}}$ *for all x, $y \in X$, and*

(ii) *for any given $\epsilon' \in (0,\epsilon)$, and all x, $y \in X$ such that $\rho(x,x') \leqslant \frac{1}{2A}(1+\rho(x,y)$,*

$$\left|\widetilde{D}_k(x,y) - \widetilde{D}_k(x',y)\right| \leqslant C\rho(x,x')^{\epsilon'}\frac{1}{(1+\rho(x,y))^{d+\epsilon}},$$

and

(iii) $\int_X \widetilde{D}_k(x,y)b(x)\,d\mu(x) = 1 = \int_X \widetilde{D}_k(x,y)b(y)\,d\mu(y)$;

and $\widetilde{D}_k(x,y)$ for $k = N+1, \cdots$ satisfy conditions (i) and (ii) of Definition 2.6 with ϵ replaced by $\epsilon' \in (0,\epsilon)$, and

$$\int_X \widetilde{D}_k(x,y)b(y)\,d\mu(y) = \int_X \widetilde{D}_k(x,y)b(x)\,d\mu(x) = 0.$$

Proof. For $k \in \mathbb{Z}_+$, let $\widetilde{D}_k(x,y) = S^{-1}[D_k^N(\cdot,y)](x)$. By (3.4), Theorem 3.3, $(S^{-1})^*(b) = b$, $S^{-1}(1) = 1$ and noting that $D_k^N(\cdot,y) \in \mathcal{G}_0(y, 2^{-k}, \epsilon, \epsilon)$ for $k = N+1, \cdots$, and for $k = 0, 1 \cdots, N$,

$$\int_X D_k^N(x,y)b(y)\, d\mu(y) = 1 = \int_X D_k^N(x,y)b(x)\, d\mu(x),$$

we can obtain all the conclusions of the theorem except (i) and (ii) and the convergence in $L^p(X)$ with $p \in (1,\infty)$ of the series in (3.26); see [44] and [16]. Let us now verify that $\widetilde{D}_k(x,y)$ for $k = 0, 1 \cdots, N$ satisfies (i) and (ii) of Theorem 3.4. It is easy to see that for all $x,\ y \in X$,

$$|D_k^N(x,y)| \leqslant C_N \frac{1}{(1+\rho(x,y))^{d+\epsilon}} \tag{3.28}$$

and for all $x,\ x' \in X$ and $\rho(x,x') \leqslant \frac{1}{2A}(1+\rho(x,y))$,

$$|D_k^N(x,y) - D_k^N(x',y)| \leqslant C_N \left(\frac{\rho(x,x')}{1+\rho(x,y)}\right)^{\epsilon} \frac{1}{(1+\rho(x,y))^{d+\epsilon}}, \tag{3.29}$$

where C_N is independent of x and x'. By (3.2), we actually have that for $k = 0, 1, \cdots, N$,

$$D_k^N = \sum_{l=0}^{k+N} D_l.$$

From this, the fact that $D_l \in \mathcal{G}_0(y, 1, \epsilon, \epsilon)$ for $l = 1 \cdots, k+N$ and Theorem 3.3, it is easy to see that we only need to verify that $S^{-1}[S_0(\cdot,y)](x)$ satisfies (i) and (ii) of Theorem 3.4. To this end, by (3.26), we first verify that for any $\epsilon' \in (0,\epsilon)$, there are $\delta \in (0,\epsilon')$ and constants $C,\ C_N > 0$ such that for all $x,\ y \in X$,

$$|R[S_0(\cdot,y)](x)| \leqslant C(2^{-\delta N} + C_N 2^{-j\delta}) \frac{1}{(1+\rho(x,y))^{d+\epsilon}} \tag{3.30}$$

and for all $x,\ x' \in X$ and $\rho(x,x') \leqslant \frac{1}{2A}(1+\rho(x,y))$,

$$\begin{aligned} |R[S_0(\cdot,y)](x) - R[S_0(\cdot,y)](x')| &\leqslant C(2^{-\delta N} + C_N^2 2^{-j\delta})\rho(x,x')^{\epsilon'} \\ &\quad\times \frac{1}{(1+\rho(x,y))^{d+\epsilon}}, \end{aligned} \tag{3.31}$$

where C_N is the same as in (3.28) and (3.29).

Similarly to the proof of Theorem 3.2, we can write

$$\begin{aligned}
&R[S_0(\cdot,y)](x)\\
&=\sum_{k=0}^{N}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\int_{Q_\tau^{k,\nu}} D_k^N(x,v)\Big[\mathcal{M}_bD_k\mathcal{M}_b(S_0(\cdot,y))(v)-D_{\tau,1}^{k,\nu}\mathcal{M}_b(S_0(\cdot,y))\Big]\,d\mu(v)\\
&\quad+\sum_{k=N+1}^{\infty}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\int_{Q_\tau^{k,\nu}} D_k^N(x,v)b(v)\\
&\quad\times\left[D_k\mathcal{M}_b(S_0(\cdot,y))(v)-D_k\mathcal{M}_b(S_0(\cdot,y))(y_\tau^{k,\nu})\right]\,d\mu(v)\\
&\quad+\sum_{k=0}^{\infty}\sum_{|l|>N} D_{k+l}\mathcal{M}_bD_k\mathcal{M}_b(f)(x)\\
&=G_1[S_0(\cdot,y)](x)+G_2[S_0(\cdot,y)](x)+R_N[S_0(\cdot,y)](x).
\end{aligned}$$

It was proved in [44] (see also [20]) that $R_N[S_0(\cdot,y)](x)$ satisfies the estimates (3.30) and (3.31). In fact, it satisfies a stronger estimate that (3.31).

We now verify that $G_2[S_0(\cdot,y)](x)$ satisfies the estimates (3.30) and (3.31). Write that

$$\begin{aligned}
G_2[S_0(\cdot,y)](x)&=\sum_{k=N+1}^{\infty}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\int_{Q_\tau^{k,\nu}}\int_X D_k^N(x,v)b(v)\\
&\quad\times\left[D_k(v,z)-D_k(y_\tau^{k,\nu},z)\right]b(z)S_0(z,y)\,d\mu(z)\,d\mu(v)\\
&=\sum_{k=N+1}^{\infty}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\int_{Q_\tau^{k,\nu}}\int_X D_k^N(x,v)b(v)\\
&\quad\times\left[D_k(v,z)-D_k(y_\tau^{k,\nu},z)\right]b(z)\left[S_0(z,y)-S_0(x,y)\right]\,d\mu(z)\,d\mu(v).
\end{aligned}$$

Since $v,\ y_\tau^{k,\nu}\in Q_\tau^{k,\nu}$, then

$$\rho(v,y_\tau^{k,\nu})\sim 2^{-j-k},\tag{3.32}$$

which together with $b\in L^\infty(X)$ in turn implies that

$$\begin{aligned}
&|G_2[S_0(\cdot,y)](x)|\\
&\leqslant C2^{-j\epsilon}\left\{\sum_{k=N+1}^{\infty}\int_X\int_{\rho(x,z)\leqslant\frac{1}{2A}(1+\rho(x,y))}\left|D_k^N(x,v)\right|\right.\\
&\quad\times\frac{2^{-k\epsilon}}{(2^{-k}+\rho(v,z))^{d+\epsilon}}\frac{\rho(x,z)^{\epsilon'}}{(1+\rho(x,y))^{d+\epsilon+\epsilon'}}\,d\mu(z)\,d\mu(v)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{k=N+1}^{\infty}\int_X\int_{\rho(x,z)>\frac{1}{2A}(1+\rho(x,y))}\left|D_k^N(x,v)\right| \\
&\times\frac{2^{-k\epsilon}}{(2^{-k}+\rho(v,z))^{d+\epsilon}}\left[|S_0(z,y)|+|S_0(x,y)|\right]d\mu(z)\,d\mu(v)\Big\} \\
&\leqslant C_N 2^{-j\epsilon}\left\{\sum_{k=N+1}^{\infty}\int_{\rho(x,z)\leqslant\frac{1}{2A}(1+\rho(x,y))}\frac{2^{-k\epsilon}}{(2^{-k}+\rho(x,z))^{d+\epsilon}}\right. \\
&\times\frac{\rho(x,z)^{\epsilon'}}{(1+\rho(x,y))^{d+\epsilon+\epsilon'}}\,d\mu(z) \\
&+\sum_{k=N+1}^{\infty}\int_{\rho(x,z)>\frac{1}{2A}(1+\rho(x,y))}\frac{2^{-k\epsilon}}{(2^{-k}+\rho(x,z))^{d+\epsilon}}\times\left[|S_0(z,y)|+|S_0(x,y)|\right]d\mu(z)\Big\} \\
&\leqslant C_N 2^{-j\epsilon}\frac{1}{(1+\rho(x,y))^{d+\epsilon}}\sum_{k=N+1}^{\infty}\left(2^{-k\epsilon}+2^{-k\epsilon'}\right) \\
&\leqslant C_N 2^{-j\epsilon}\frac{1}{(1+\rho(x,y))^{d+\epsilon}},
\end{aligned}
$$

where $\epsilon' \in (0,\epsilon)$ and we omit some computation similar to the proof of (3.18). This verifies $G_2[S_0(\cdot,y)](x)$ satisfies (3.30).

We now show $G_2[S_0(\cdot,y)](x)$ satisfies (3.31). To this end, set

$$
\begin{aligned}
W_1 &= \left\{v\in X:\ \rho(x,x')\leqslant\frac{1}{2A}(2^{-k}+\rho(x,v))\right\}, \\
W_2 &= \left\{v\in X:\ \rho(x,x')>\frac{1}{2A}(2^{-k}+\rho(x,v))\right\}, \\
W_3 &= \left\{z\in X:\ \rho(x,z)\leqslant\frac{1}{2A}(1+\rho(x,y))\right\}, \\
W_4 &= \left\{z\in X:\ \rho(x,z)>\frac{1}{2A}(1+\rho(x,y))\right\}, \\
W_5 &= \left\{z\in X:\ \rho(x',z)\leqslant\frac{1}{2A}(1+\rho(x,y))\right\}
\end{aligned}
$$

and

$$
W_6 = \left\{z\in X:\ \rho(x',z)>\frac{1}{2A}(1+\rho(x,y))\right\}.
$$

Write that

$$
\begin{aligned}
&G_2[S_0(\cdot,y)](x)-G_2[S_0(\cdot,y)](x') \\
&=\sum_{k=N+1}^{\infty}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\int_{Q_\tau^{k,\nu}}\int_X\left[D_k^N(x,v)-D_k^N(x',v)\right]b(v) \\
&\times\left[D_k(v,z)-D_k(y_\tau^{k,\nu},z)\right]b(z)S_0(z,y)\,d\mu(z)\,d\mu(v)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{k=N+1}^{\infty} \sum_{\tau\in I_k} \sum_{\nu=1}^{N(k,\tau)} \int_{Q_\tau^{k,\nu}\cap W_1} \int_{X\cap W_3} \left[D_k^N(x,v) - D_k^N(x',v)\right] b(v) \\
&\quad \times \left[D_k(v,z) - D_k(y_\tau^{k,\nu},z)\right] b(z) \left[S_0(z,y) - S_0(x,y)\right] d\mu(z)\, d\mu(v) \\
&\quad + \sum_{k=N+1}^{\infty} \sum_{\tau\in I_k} \sum_{\nu=1}^{N(k,\tau)} \int_{Q_\tau^{k,\nu}\cap W_1} \int_{X\cap W_4} \cdots \\
&\quad + \sum_{k=N+1}^{\infty} \sum_{\tau\in I_k} \sum_{\nu=1}^{N(k,\tau)} \int_{Q_\tau^{k,\nu}\cap W_2} \int_{X\cap W_3} D_k^N(x,v) b(v) \left[D_k(v,z) - D_k(y_\tau^{k,\nu},z)\right] \\
&\quad \times b(z) \left[S_0(z,y) - S_0(x,y)\right] d\mu(z)\, d\mu(v) \\
&\quad + \sum_{k=N+1}^{\infty} \sum_{\tau\in I_k} \sum_{\nu=1}^{N(k,\tau)} \int_{Q_\tau^{k,\nu}\cap W_2} \int_{X\cap W_4} \cdots \\
&\quad - \sum_{k=N+1}^{\infty} \sum_{\tau\in I_k} \sum_{\nu=1}^{N(k,\tau)} \int_{Q_\tau^{k,\nu}\cap W_2} \int_{X\cap W_3} D_k^N(x',v) b(v) \left[D_k(v,z) - D_k(y_\tau^{k,\nu},z)\right] \\
&\quad \times b(z) \left[S_0(z,y) - S_0(x',y)\right] d\mu(z)\, d\mu(v) \\
&\quad - \sum_{k=N+1}^{\infty} \sum_{\tau\in I_k} \sum_{\nu=1}^{N(k,\tau)} \int_{Q_\tau^{k,\nu}\cap W_2} \int_{X\cap W_4} \cdots \\
&= \sum_{i=1}^{6} H_i.
\end{aligned}
$$

By (3.32), $b \in L^\infty(X)$ and the proof of (3.13), we have

$$
\begin{aligned}
|H_1| &\leqslant C_N 2^{-j\epsilon} \sum_{k=N+1}^{\infty} \int_{X\cap W_1} \int_{X\cap W_3} \frac{\rho(x,x')^{\epsilon'} 2^{-k\epsilon}}{(2^{-k} + \rho(x,v))^{d+\epsilon+\epsilon'}} \\
&\quad \times \frac{2^{-k\epsilon}}{(2^{-k} + \rho(v,z))^{d+\epsilon}} \frac{\rho(x,z)^{\epsilon''}}{(1 + \rho(x,y))^{d+\epsilon+\epsilon''}}\, d\mu(z)\, d\mu(v) \\
&\leqslant C_N 2^{-j\epsilon} \frac{\rho(x,x')^{\epsilon'}}{(1+\rho(x,y))^{d+\epsilon+\epsilon''}} \sum_{k=N+1}^{\infty} 2^{k\epsilon'} \int_X \frac{2^{-k\epsilon}}{(2^{-k} + \rho(x,z))^{d+\epsilon}} \rho(x,z)^{\epsilon''}\, d\mu(z) \\
&\leqslant C_N 2^{-j\epsilon} \frac{\rho(x,x')^{\epsilon'}}{(1+\rho(x,y))^{d+\epsilon+\epsilon''}} \sum_{k=N+1}^{\infty} 2^{-k(\epsilon''-\epsilon')} \\
&\leqslant C_N 2^{-j\epsilon} \frac{\rho(x,x')^{\epsilon'}}{(1+\rho(x,y))^{d+\epsilon}},
\end{aligned} \tag{3.33}
$$

where $\epsilon' \in (0,\epsilon)$ and $\epsilon'' \in (\epsilon',\epsilon)$, which is a desired estimate.

Again, (3.32), $b \in L^\infty(X)$ and the proof of (3.13) imply that

$$|H_2| \leqslant C_N 2^{-j\epsilon} \sum_{k=N+1}^{\infty} \int_{X\cap W_1} \int_{X\cap W_4} \frac{\rho(x,x')^{\epsilon'} 2^{-k\epsilon}}{(2^{-k}+\rho(x,v))^{d+\epsilon+\epsilon'}}$$
$$\times \frac{2^{-k\epsilon}}{(2^{-k}+\rho(v,z))^{d+\epsilon}} [|S_0(z,y)| + |S_0(x,y)|]\, d\mu(z)\, d\mu(v)$$
$$\leqslant C_N 2^{-j\epsilon} \rho(x,x')^{\epsilon'} \sum_{k=N+1}^{\infty} 2^{k\epsilon'} \int_{X\cap W_4} \frac{2^{-k\epsilon}}{(2^{-k}+\rho(x,z))^{d+\epsilon}}$$
$$\times [|S_0(z,y)| + |S_0(x,y)|]\, d\mu(z)$$
$$\leqslant C_N 2^{-j\epsilon} \rho(x,x')^{\epsilon'} \frac{1}{(1+\rho(x,y))^{d+\epsilon}} \sum_{k=N+1}^{\infty} 2^{-k(\epsilon''-\epsilon')}$$
$$\leqslant C_N 2^{-j\epsilon} \rho(x,x')^{\epsilon'} \frac{1}{(1+\rho(x,y))^{d+\epsilon}},$$

where we chose ϵ', ϵ'' as in the estimate of (3.33), which is also a desired estimate.

From the estimate (3.32), $b \in L^\infty(X)$ and the proof of (3.13), it follows that

$$|H_3| \leqslant C 2^{-j\epsilon} \sum_{k=N+1}^{\infty} \int_{X\cap W_2} \int_{X\cap W_3} |D_k^N(x,v)| \frac{2^{-k\epsilon}}{(2^{-k}+\rho(v,z))^{d+\epsilon}}$$
$$\times \frac{\rho(x,z)^{\epsilon''}}{(1+\rho(x,y))^{d+\epsilon+\epsilon''}}\, d\mu(z)\, d\mu(v)$$
$$\leqslant C_N 2^{-j\epsilon} \sum_{k=N+1}^{\infty} \int_{X\cap W_3} \frac{2^{-k\epsilon}}{(2^{-k}+\rho(x,z))^{d+\epsilon}} \frac{\rho(x,z)^{\epsilon''}}{(1+\rho(x,y))^{d+\epsilon+\epsilon''}}\, d\mu(z)$$
$$\leqslant C_N 2^{-j\epsilon} \frac{\rho(x,x')^{\epsilon'}}{(1+\rho(x,y))^{d+\epsilon+\epsilon''}} \sum_{k=N+1}^{\infty} 2^{-k(\epsilon''-\epsilon')}$$
$$\leqslant C_N 2^{-j\epsilon} \frac{\rho(x,x')^{\epsilon'}}{(1+\rho(x,y))^{d+\epsilon}},$$

and

$$|H_4| \leqslant C 2^{-j\epsilon} \sum_{k=N+1}^{\infty} \int_{X\cap W_2} \int_{X\cap W_4} |D_k^N(x,v)| \frac{2^{-k\epsilon}}{(2^{-k}+\rho(v,z))^{d+\epsilon}}$$
$$\times [|S_0(z,y)| + |S_0(x,y)|]\, d\mu(z)\, d\mu(v)$$
$$\leqslant C_N 2^{-j\epsilon} \rho(x,x')^{\epsilon'} \sum_{k=N+1}^{\infty} 2^{k\epsilon'} \int_{X\cap W_4} \frac{2^{-k\epsilon}}{(2^{-k}+\rho(x,z))^{d+\epsilon}}$$
$$\times [|S_0(z,y)| + |S_0(x,y)|]\, d\mu(z)$$

$$\leqslant C_N 2^{-j\epsilon}\rho(x,x')^{\epsilon'}\frac{1}{(1+\rho(x,y))^{d+\epsilon}}\sum_{k=N+1}^{\infty}\left(2^{-k(\epsilon-\epsilon')+2^{-k(\epsilon''-\epsilon')}}\right)$$
$$\leqslant C_N 2^{-j\epsilon}\rho(x,x')^{\epsilon'}\frac{1}{(1+\rho(x,y))^{d+\epsilon}},$$

where we chose ϵ', ϵ'' as in the estimate of (3.33), which are the desired estimates.

Similarly to the estimates for H_3 and H_4, we can verify that

$$|H_5|+|H_6|\leqslant C_N 2^{-j\epsilon}\rho(x,x')^{\epsilon'}\frac{1}{(1+\rho(x',y))^{d+\epsilon}}.$$

Since we have $\rho(x,x')\leqslant\frac{1}{2A}(1+\rho(x,y))$, we then deduce that $1+\rho(x',y)\sim 1+\rho(x,y)$. From this, we can also deduce the desired estimates for H_5 and H_6. Thus, $G_2[S_0(\cdot,y)](x)$ satisfies (3.31).

The proof for that $G_1[S_0(\cdot,y)](x)$ satisfies (3.30) and (3.31) is quite similar to that for $G_2[S_0(\cdot,y)](x)$ by using that fact that $b\in SPF(X)$; see also the proof of Theorem 3.2. We leave the details to the reader. Thus, (3.30) and (3.31) holds.

Note that $R^*(b)=0$ implies

$$\int_X R[S_0(\cdot,y)](x)b(x)\,d\mu(x)=0. \tag{3.34}$$

Thus, (3.30), (3.31) and (3.34) indicate that $R[S_0(\cdot,y)](x)\in\mathcal{G}_0(y,1,\epsilon',\epsilon-\epsilon')$ with

$$\|R[S_0(\cdot,y)]\|_{\mathcal{G}(y,1,\epsilon',\epsilon-\epsilon')}\leqslant C\left(2^{-\delta N}+C_N 2^{-j\delta}\right).$$

By Theorem 3.2 and Lemma 2.2, we then have that for any $m\in\mathbb{N}$,

$$R^m[S_0(\cdot,y)]\in\mathcal{G}_0(y,1,\epsilon',\epsilon-\epsilon') \tag{3.35}$$

and

$$\|R^m[S_0(\cdot,y)]\|_{\mathcal{G}(y,1,\epsilon',\epsilon-\epsilon')}\leqslant C_9^m\left(2^{-\delta N}+C_N 2^{-j\delta}\right)^m.$$

Form this and (3.26), it follows that if we choose $N,\ j\in\mathbb{N}$ large enough such that

$$C_9\left(2^{-\delta N}+C_N 2^{-j\delta}\right)<1,$$

then $S^{-1}[S_0(\cdot,y)](x)$ satisfies (i) and (ii) of Theorem 3.4.

Let us finally show that the series in (3.26) converge in the norm of $L^p(X)$ for $p \in (1,\infty)$. To this end, for $L > N$, we write

$$\begin{aligned}
&\sum_{k=0}^{N}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\int_{Q_\tau^{k,\nu}} \widetilde{D}_k(x,y)b(y)\,d\mu(y)D_{\tau,1}^{k,\nu}\mathcal{M}_b(f)\\
&\quad+\sum_{k=N+1}^{L}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\int_{Q_\tau^{k,\nu}} \widetilde{D}_k(x,y)b(y)\,d\mu(y)D_k\mathcal{M}_b(f)(y_\tau^{k,\nu})\\
&\quad= S^{-1}\Bigg[\sum_{k=0}^{N}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\int_{Q_\tau^{k,\nu}} D_k^N(\cdot,y)b(y)\,d\mu(y)D_{\tau,1}^{k,\nu}\mathcal{M}_b(f)\\
&\quad+\sum_{k=N+1}^{L}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\int_{Q_\tau^{k,\nu}} D_k^N(\cdot,y)b(y)\,d\mu(y)D_k\mathcal{M}_b(f)(y_\tau^{k,\nu})\Bigg](x)\\
&\quad= S^{-1}\Bigg\{S(f)(\cdot)-\sum_{k=L+1}^{\infty}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\int_{Q_\tau^{k,\nu}} D_k^N(\cdot,y)b(y)\,d\mu(y)D_k\mathcal{M}_b(f)(y_\tau^{k,\nu})\Bigg\}(x)\\
&\quad= f(x)-\lim_{m\to\infty}R^m(f)(x)\\
&\quad- S^{-1}\Bigg\{\sum_{k=L+1}^{\infty}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\int_{Q_\tau^{k,\nu}} D_k^N(\cdot,y)b(y)\,d\mu(y)D_k\mathcal{M}_b(f)(y_\tau^{k,\nu})\Bigg\}(x).
\end{aligned}$$

To show the theorem, we need to show that $R^m(f)(x)$ and

$$S^{-1}\Bigg\{\sum_{k=L+1}^{\infty}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\int_{Q_\tau^{k,\nu}} D_k^N(\cdot,y)b(y)\,d\mu(y)D_k\mathcal{M}_b(f)(y_\tau^{k,\nu})\Bigg\}(x)$$

converge to zero in the norm of $L^p(X)$ for $p \in (1,\infty)$ as m and L goes to ∞. By Theorem 3.2 and the Tb theorem in [6] (see also [4]), we have that for $p \in (1,\infty)$ and all $f \in L^p(X)$,

$$\|R^m(f)\|_{L^p(X)} \leqslant C_{10}^m(C_N 2^{-j\delta}+C2^{-N\delta})^m\|f\|_{L^p(X)},$$

where C_{10} and C_N are independent of f and m. This shows $\lim_{m\to\infty} R^m(f)=0$ in $L^p(X)$ for $p \in (1,\infty)$ and fixed large positive integers j and N. It remains to show that for $p \in (1,\infty)$,

$$\lim_{L\to\infty}\Bigg\|\sum_{k=L+1}^{\infty}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\int_{Q_\tau^{k,\nu}} D_k^N(\cdot,y)b(y)\,d\mu(y)D_k\mathcal{M}_b(f)(y_\tau^{k,\nu})\Bigg\|_{L^p(X)} = 0. \quad (3.36)$$

Let $1/p+1/q=1$. We write

$$\begin{aligned}&\left\|\sum_{k=L+1}^{\infty}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\int_{Q_\tau^{k,\nu}}D_k^N(\cdot,y)b(y)\,d\mu(y)D_k\mathcal{M}_b(f)(y_\tau^{k,\nu})\right\|_{L^p(X)}\\&=\sup_{\|g\|_{L^q(X)}\leqslant 1}\left|\sum_{k=L+1}^{\infty}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\int_{Q_\tau^{k,\nu}}(D_k^N)^*(g)(y)b(y)\,d\mu(y)D_k\mathcal{M}_b(f)(y_\tau^{k,\nu})\right|\\&=\sup_{\|g\|_{L^q(X)}\leqslant 1}\sum_{k=L+1}^{\infty}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\int_{Q_\tau^{k,\nu}}\left|(D_k^N)^*(g)(y)D_k\mathcal{M}_b(f)(y_\tau^{k,\nu})\right|\,d\mu(y),\end{aligned}$$

where we used the fact that $b\in L^\infty(X)$.

Let $\{\widetilde{D}_l\}_{l=0}^\infty$ be the same as in Lemma 2.3. It was proved that in Lemma 3.1 of [44] (see also [16]) that we have that for any $\epsilon''\in(0,\epsilon')$, all $y\in Q_\tau^{k,\nu}$, all $z\in X$ and $l\in\mathbb{Z}_+$,

$$|(D_k^N)^*\widetilde{D}_l(y,z)|\leqslant C2^{-|k-l|\epsilon''}\frac{2^{-(k\wedge l)\epsilon'}}{(2^{-(k\wedge l)}+\rho(y,z))^{d+\epsilon'}},$$

which also holds if we replace $(D_k^N)^*$ by D_k with $k=N+1,\cdots$. From this, $b\in L^\infty(X)$, Lemma 2.3, Lemma 2.4, Lemma 2.7, Lemma 2.8 and the Hölder inequality, it follows that

$$\begin{aligned}&\sum_{k=L+1}^{\infty}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\int_{Q_\tau^{k,\nu}}\left|(D_k^N)^*(g)(y)D_k\mathcal{M}_b(f)(y_\tau^{k,\nu})\right|\,d\mu(y)\\&\leqslant C\sum_{k=L+1}^{\infty}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\int_{Q_\tau^{k,\nu}}\left[\sum_{l=0}^{\infty}2^{-|k-l|\epsilon''}M(D_l(g))(y)\right]\\&\quad\times\left[\sum_{l=0}^{\infty}2^{-|k-l|\epsilon''}M(D_l\mathcal{M}_b(f))(y)\right]d\mu(y)\\&\leqslant C\int_X\left\{\sum_{k=L+1}^{\infty}\left[\sum_{l=0}^{\infty}2^{-|k-l|\epsilon''}M(D_l(g))(y)\right]^2\right\}^{1/2}\\&\quad\times\left\{\sum_{k=L+1}^{\infty}\left[\sum_{l=0}^{\infty}2^{-|k-l|\epsilon''}M(D_l\mathcal{M}_b(f))(y)\right]^2\right\}^{1/2}d\mu(y)\end{aligned}$$

$$\leqslant C\int_X \left\{\sum_{k=L+1}^{\infty}\sum_{l=0}^{\infty}2^{-|k-l|\epsilon''}[M(D_l(g))(y)]^2\right\}^{1/2}$$

$$\times\left\{\sum_{k=L+1}^{\infty}\sum_{l=0}^{\infty}2^{-|k-l|\epsilon''}[M(D_l\mathcal{M}_b(f))(y)]^2\right\}^{1/2} d\mu(y)$$

$$\leqslant C\left\|\left\{\sum_{l=0}^{\infty}[M(D_l(g))]^2\right\}^{1/2}\right\|_{L^q(X)}$$

$$\times\left\|\left\{\sum_{k=L+1}^{\infty}\sum_{l=0}^{\infty}2^{-|k-l|\epsilon''}[M(D_l\mathcal{M}_b(f))]^2\right\}^{1/2}\right\|_{L^p(X)}$$

$$\leqslant C\|g\|_{L^q(X)}\left\|\left\{\sum_{k=L+1}^{\infty}\sum_{l=0}^{\infty}2^{-|k-l|\epsilon''}[M(D_l\mathcal{M}_b(f))]^2\right\}^{1/2}\right\|_{L^p(X)}.$$

Thus, by Lemma 2.7 and Lemma 2.8 again, we have

$$\left\|\sum_{k=L+1}^{\infty}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\int_{Q_\tau^{k,\nu}} D_k^N(\cdot,y)b(y)\,d\mu(y)D_k\mathcal{M}_b(f)(y_\tau^{k,\nu})\right\|_{L^p(X)}$$

$$\leqslant C\left\|\left\{\sum_{k=L+1}^{\infty}\sum_{l=0}^{\infty}2^{-|k-l|\epsilon''}[M(D_l\mathcal{M}_b(f))]^2\right\}^{1/2}\right\|_{L^p(X)}$$

$$\leqslant C2^{-\epsilon'' L/2}\left\|\left\{\sum_{l=0}^{L/2}[M(D_l\mathcal{M}_b(f))]^2\right\}^{1/2}\right\|_{L^p(X)}$$

$$+C\left\|\left\{\sum_{l=L/2+1}^{\infty}[M(D_l\mathcal{M}_b(f))]^2\right\}^{1/2}\right\|_{L^p(X)}$$

$$\leqslant C2^{-\epsilon'' L/2}\|f\|_{L^p(X)}+C\left\|\left\{\sum_{l=L/2+1}^{\infty}|D_l\mathcal{M}_b(f)|^2\right\}^{1/2}\right\|_{L^p(X)},$$

which converges to 0 as L tends to ∞. That is, (3.36) holds and we complete the proof of Theorem 3.4. ■

Remark 3.1. Similar to the case of the continuous Calderón reproducing formulae, by rearranging the order of the approximation to the identity, without loss of generality and for the sake of simplicity, in what follows, we can take $N = 0$ in Theorem 3.4.

Remark 3.2. above inhomogeneous discrete Calderón reproducing formulae indicates an essential difference from the homogeneous Calderón reproducing formulae in [17]; see also [20, 16]. Also, (ii) of Theorem 3.4 indicates a difference between the inhomogeneous discrete Calderón reproducing formulae and the inhomogeneous continuous Calderón reproducing formulae; see Lemma 2.3 (or [44]). However, if the approximation to the identity $\{S_k\}_{k\in\mathbb{Z}_+}$ is an approximation to the identity of order $\epsilon \in (0, \theta]$ with compact support as in Remark 2.1, then (ii) of Theorem 3.4 can be improved into

(ii)' for any given $\epsilon' \in (0, \epsilon)$, all $x,\ x' \in X$ and all $y \in X$ satisfying $\rho(x, x') \leqslant \frac{1}{2A}(1 + \rho(x, y))$,

$$|\widetilde{D}_k(x, y) - \widetilde{D}_k(x', y)| \leqslant C\left(\frac{\rho(x, x')}{1 + \rho(x, y)}\right)^{\epsilon'} \frac{1}{(1 + \rho(x, y))^{d+\epsilon}}.$$

To see this, we only need to re-estimate H_2, H_4 and H_6 in the proof of Theorem 3.4. For H_2, since $\operatorname{supp} D_k(\cdot, z) \subset B(z, C2^{-k})$, if $H_2 \neq 0$, then $\rho(v, z) \leqslant C2^{-k}$ or $\rho(y_\tau^{k,\nu}, z) \leqslant C2^{-k}$ and $v \in Q_\tau^{k,\nu}$; thus, we always have $\rho(v, z) \leqslant C2^{-k}$, and therefore

$$2^{-k} + \rho(x, v) \geqslant 2^{-k} \geqslant C(2^{-k} + \rho(v, z)),$$

which implies

$$\begin{aligned}
&\sum_{\tau\in I_k} \sum_{\nu=1}^{N(k,\tau)} \int_{Q_\tau^{k,\nu}\cap W_1} |D_k^N(x, v) - D_k^N(x', v)|\, |D_k(v, z) - D_k(y_\tau^{k,\nu}, z)|\, d\mu(v) \\
&\quad \leqslant C_N 2^{-j\epsilon} \int_{\rho(x,v)\geqslant \frac{1}{2A}\rho(x,z)} \frac{2^{-k\epsilon}}{(2^{-k} + \rho(x, v))^{d+\epsilon+\epsilon'}} \frac{2^{-k\epsilon}}{(2^{-k} + \rho(v, z))^{d+\epsilon}} \\
&\quad\quad + C \int_{\rho(x,v)< \frac{1}{2A}\rho(x,z)} \frac{2^{-k\epsilon}}{(2^{-k} + \rho(x, v))^{d+\epsilon}} \frac{2^{-k\epsilon}}{(2^{-k} + \rho(v, z))^{d+\epsilon+\epsilon'}} \\
&\quad \leqslant C \frac{2^{-k\epsilon}}{(2^{-k} + \rho(x, z))^{d+\epsilon+\epsilon'}}.
\end{aligned}$$

Replacing this estimate into that of H_2 in the proof of Theorem 3.4, we can obtain a desired estimate for H_2. The same technique works for the estimates of H_4 and H_6 in the proof of Theorem 3.4. We omit the details.

By a duality argument, Theorem 3.4 tells us the following inhomogeneous discrete Calderón reproducing formulae associated to a given special para-accretive function in distribution spaces.

Theorem 3.5. *Let b be a special para-accretive function. Suppose that $\{S_k\}_{k\in\mathbb{Z}_+}$ is an approximation to the identity of order $\epsilon \in (0, \theta]$ as in Definition 2.6 and $\{D_k\}_{k\in\mathbb{Z}_+}$ is as in Section 2. Then there exists a family of linear operators $\{\widetilde{D}_k\}_{k=0}^\infty$ such that for any fixed $y_\tau^{k,\nu} \in Q_\tau^{k,\nu}$ with $k \in \mathbb{N}$, $\tau \in I_k$ and $\nu \in \{1, \cdots, N(k, \tau)\}$*

and all $f \in \left(\mathring{\mathcal{G}}(\beta_1,\gamma_1)\right)'$ *with* $0<\beta_1,\ \gamma_1<\epsilon$,

$$f(x)=\sum_{\tau\in I_0}\sum_{\nu=1}^{N(0,\tau)} b(x)\left[\frac{1}{b(Q_\tau^{0,\nu})}\int_{Q_\tau^{0,\nu}} D_0(x,u)b(u)\,d\mu(u)\right]\int_{Q_\tau^{0,\nu}} b(y)\widetilde{E}_0(f)(y)\,d\mu(y)$$
$$+\sum_{k=1}^{\infty}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)} b(x)D_k(x,y_\tau^{k,\nu})\int_{Q_\tau^{k,\nu}} b(y)\widetilde{E}_k(f)(y)\,d\mu(y),$$

where the series converge in $\left(\mathring{\mathcal{G}}(\beta_1',\gamma_1')\right)'$ *with* $\beta_1<\beta_1'<\epsilon$ *and* $\gamma_1<\gamma_1'<\epsilon$. *Moreover, there is a constant* $C>0$ *such that* $\widetilde{E}_0(x,y)$, *the kernel of the linear operator* $\widetilde{E}_0$ *satisfies*

(i) $\left|\widetilde{E}_0(x,y)\right|\leqslant C\frac{1}{(1+\rho(x,y))^{d+\epsilon}}$ *for all* $x,\ y\in X$, *and*

(ii) *for any given* $\epsilon'\in(0,\epsilon)$,

$$\left|\widetilde{E}_0(x,y)-\widetilde{E}_0(x,y')\right|\leqslant C\rho(y,y')^{\epsilon'}\frac{1}{(1+\rho(x,y))^{d+\epsilon}}$$

for all $x,\ y\in X$ *such that* $\rho(y,y')\leqslant\frac{1}{2A}(1+\rho(x,y))$, *and*

(iii) $\int_X \widetilde{E}_0(x,y)b(x)\,d\mu(x)=1=\int_X \widetilde{E}_0(x,y)b(y)\,d\mu(y)$;

and $\widetilde{E}_k(x,y)$, *the kernel of the linear operator* $\widetilde{E}_k$ *for* $k\in\mathbb{N}$ *satisfies the conditions (i) and (iii) of Definition 2.6 with* ϵ *replaced by* $\epsilon'\in(0,\epsilon)$, *and*

$$\int_X \widetilde{E}_k(x,y)b(y)\,d\mu(y)=\int_X \widetilde{E}_k(x,y)b(x)\,d\mu(x)=0.$$

By an argument similar to the proofs of Theorem 3.4 and Theorem 3.5, we can prove the following several relative theorems. We only state them and leave the details to the reader.

Theorem 3.6. *With all the notation same as in Theorem 3.5, then for all* $f\in\mathcal{G}(\beta_1,\gamma_1)$ *with* $0<\beta_1,\ \gamma_1<\epsilon$,

$$f(x)=\sum_{\tau\in I_0}\sum_{\nu=1}^{N(0,\tau)}\int_{Q_\tau^{0,\nu}} D_0(x,y)b(y)\,d\mu(y)\left[\frac{1}{b(Q_\tau^{0,\nu})}\int_{Q_\tau^{0,\nu}} b(u)\widetilde{E}_0\mathcal{M}_b(f)(u)\,d\mu(u)\right]$$
$$+\sum_{k=1}^{\infty}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\int_{Q_\tau^{k,\nu}} D_k(x,y)b(y)\,d\mu(y)\widetilde{E}_k\mathcal{M}_b(f)(y_\tau^{k,\nu}),$$

where the series converge in the norms of $L^p(X)$, $1<p<\infty$, *and* $\mathcal{G}(\beta_1',\gamma_1')$ *for* $0<\beta_1'<\beta_1$ and $0<\gamma_1'<\gamma_1$.

Theorem 3.7. *With all the notation same as in Theorem 3.4 but with $N = 0$, then for all $f \in \left(\mathring{\mathcal{G}}(\beta_1, \gamma_1)\right)'$ with $0 < \beta_1,\ \gamma_1 < \epsilon$,*

$$f(x) = \sum_{\tau \in I_0} \sum_{\nu=1}^{N(0,\tau)} b(x) \left[\frac{1}{b(Q_\tau^{0,\nu})} \int_{Q_\tau^{0,\nu}} \widetilde{D}_0(x,u) b(u)\, d\mu(u)\right] \int_{Q_\tau^{0,\nu}} b(y) D_0(f)(y)\, d\mu(y)$$
$$+ \sum_{k=1}^{\infty} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} b(x) \widetilde{D}_k(x, y_\tau^{k,\nu}) \int_{Q_\tau^{k,\nu}} b(y) D_k(f)(y)\, d\mu(y),$$

where the series converge in $\left(\mathring{\mathcal{G}}(\beta_1', \gamma_1')\right)'$ with $\beta_1 < \beta_1' < \epsilon$ and $\gamma_1 < \gamma_1' < \epsilon$.

By the definition of the space $b\mathcal{G}(\beta_1, \gamma_1)$, Theorem 3.4 and Theorem 3.6, we can obtain the following theorem.

Theorem 3.8. *With all the notation same as in Theorem 3.4 with $N = 0$ and Theorem 3.5, then for all $f \in b\mathcal{G}(\beta_1, \gamma_1)$ with $0 < \beta_1,\ \gamma_1 < \epsilon$,*

$$f(x) = \sum_{k=0}^{N} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} b(x) \int_{Q_\tau^{k,\nu}} \widetilde{D}_k(x,y) b(y)\, d\mu(y) D_{\tau,1}^{k,\nu}(f)$$
$$+ \sum_{k=N+1}^{\infty} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} b(x) \int_{Q_\tau^{k,\nu}} \widetilde{D}_k(x,y) b(y)\, d\mu(y) D_k(f)(y_\tau^{k,\nu})$$
$$= \sum_{\tau \in I_0} \sum_{\nu=1}^{N(0,\tau)} b(x) \int_{Q_\tau^{0,\nu}} D_0(x,y) b(y)\, d\mu(y) \left[\frac{1}{b(Q_\tau^{0,\nu})} \int_{Q_\tau^{0,\nu}} b(u) \widetilde{E}_0(f)(u)\, d\mu(u)\right]$$
$$+ \sum_{k=1}^{\infty} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} b(x) \int_{Q_\tau^{k,\nu}} D_k(x,y) b(y)\, d\mu(y) \widetilde{E}_k(f)(y_\tau^{k,\nu}),$$

where the series converge in the norms of $L^p(X)$, $1 < p < \infty$, and $b\mathcal{G}(\beta_1', \gamma_1')$ for $0 < \beta_1' < \beta_1$ and $0 < \gamma_1' < \gamma_1$.

From (2.3), Theorem 3.5 and Theorem 3.7, it follows the following conclusions.

Theorem 3.9. *With all the notation same as in Theorem 3.4 with* $N = 0$ *and Theorem 3.5, then for all* $f \in \left(b\mathring{\mathcal{G}}(\beta_1, \gamma_1)\right)'$ *with* $0 < \beta_1,\ \gamma_1 < \epsilon$,

$$\begin{aligned} f(x) &= \sum_{\tau \in I_0} \sum_{\nu=1}^{N(0,\tau)} \left[\frac{1}{b(Q_\tau^{0,\nu})} \int_{Q_\tau^{0,\nu}} D_0(x,u) b(u)\, d\mu(u)\right] \int_{Q_\tau^{0,\nu}} b(y) \widetilde{E}_0 \mathcal{M}_b(f)(y)\, d\mu(y) \\ &\quad + \sum_{k=1}^{\infty} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} D_k(x, y_\tau^{k,\nu}) \int_{Q_\tau^{k,\nu}} b(y) \widetilde{E}_k \mathcal{M}_b(f)(y)\, d\mu(y) \\ &= \sum_{\tau \in I_0} \sum_{\nu=1}^{N(0,\tau)} \left[\frac{1}{b(Q_\tau^{0,\nu})} \int_{Q_\tau^{0,\nu}} \widetilde{D}_0(x,u) b(u)\, d\mu(u)\right] \int_{Q_\tau^{0,\nu}} b(y) D_0 \mathcal{M}_b(f)(y)\, d\mu(y) \\ &\quad + \sum_{k=1}^{\infty} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} \widetilde{D}_k(x, y_\tau^{k,\nu}) \int_{Q_\tau^{k,\nu}} b(y) D_k \mathcal{M}_b(f)(y)\, d\mu(y), \end{aligned}$$

where the series converge in $\left(b\mathring{\mathcal{G}}(\beta_1', \gamma_1')\right)'$ *with* $\beta_1 < \beta_1' < \epsilon$ and $\gamma_1 < \gamma_1' < \epsilon$.

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