

ON THE ZEROS OF FUNCTIONS FROM THE EXTENDED SELBERG CLASS OF DEGREE 0

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Abstract: We give an improved upper bound for the error term in the Riemann-von Mangoldt formula for functions of degree 0 in the extended Selberg class. The bound is uniform for functions with no zeros in a certain half-plane. The first such bound was given by J. Kaczorowski and A. Perelli [3] in the proof that the extended Selberg class is a semigroup with factorization.

Keywords: Selber class, zeros of L -functions, Riemann-von Mangoldt formula.

1. Introduction

The Selberg class, denoted \mathcal{S} , and the extended Selberg class, \mathcal{S}^\sharp consist of special Dirichlet series with functional equation. Both have been extensively studied in the recent years. We refer the reader to [2] or [1] for definitions and basic properties of \mathcal{S} and \mathcal{S}^\sharp . Here we only recall them in the context of the class \mathcal{S}_0^\sharp , i.e. the functions of degree 0 in \mathcal{S}^\sharp . It is well known (cf. [1]) that $F \in \mathcal{S}_0^\sharp$ if and only if it is of the form

$$F(s) = \sum_{n|q_F} a_F(n)n^{-s}$$

with $a_F(n) = \frac{\omega n}{\sqrt{q_F}} \overline{a_F\left(\frac{q_F}{n}\right)}$, $n | q_F$, with q_F a positive integer and ω a complex number, $|\omega| = 1$, F not identically zero. The number q_F is uniquely determined and called the conductor of F . There exists a real number σ_F such that $F(s) \neq 0$ in the half-plane $\operatorname{Re} s > \sigma_F$. We denote by $\Omega(n)$ the number of prime divisors of n , counted with multiplicities.

We give an upper bound for the error term in the Riemann-von Mangoldt formula for F in terms of the conductor of F , uniform for all F with no zeros in a given half-plane. Let $N_F(T)$ denote the number of zeros ρ of F with $|\operatorname{Im} \rho| \leq T$ (all zeros are non-trivial for $F \in \mathcal{S}_0^\sharp$).

Theorem. *If $F \in \mathcal{S}_0^\sharp$ and $F(s) \neq 0$ in the half-plane $\operatorname{Re} s > \sigma_F$, then*

$$N_F(T) = \frac{T}{\pi} \log q_F + O_{\sigma_F}(\log^2 q_F (\log \log q_F)^2)$$

uniformly for $T \geq 2$ and $F \in \mathcal{S}_0^\sharp$ with $a(1) = 1$, $q_f \geq 2$.

This is an improvement over the bound given by J. Kaczorowski and A. Perelli [3], who showed the above with $O_{\sigma_F}(\log^6 q_F)$. It was the main part of their proof that \mathcal{S}^\sharp is a semigroup with factorization. The improvement is based on a better upper bound for the quantity $a(n, \delta_0)$ defined for $\delta_0 \geq 1$ and $n \in \mathbb{N}$ by $a(1, \delta_0) = \delta_0$ and

$$a(n, \delta_0) = \delta_0 + \sum_{l=2}^{\Omega(n)} \frac{1}{l} \sum_{\substack{n_1 \cdots n_l = n \\ n_1, \dots, n_l \geq 2}} a(n_1, \delta_0) \cdots a(n_l, \delta_0), \quad n \geq 2,$$

an empty sum being equal to 0. It was shown in [3] that $a(n, \delta_0) \leq \delta_0^{\Omega(n)} 2^{\Omega(n)^3}$, $n \geq 2$. We show

Lemma. *If $\delta_0 \geq 1$ and $n \geq 2$, then we have*

$$\delta_0^{\Omega(n)} \exp\left(\frac{1}{7}\Omega(n) - 1\right) < a(n, \delta_0) < \delta_0^{\Omega(n)} \exp(3\Omega(n) \log \Omega(n)).$$

The proposition follows from the improved upper bound in the lemma in much the same way as in [3] so in fact we only need to prove the lemma. The lower bound shows that the upper bound cannot be made much lower. In fact with the argument given here and some computations it can be shown that

$$a(n, \delta_0) > \delta_0^{\Omega(n)} \exp(\Omega(n) \log \Omega(n))$$

if $n \geq 60$ and n is a product of distinct primes. On the other hand we have

$$a(n, \delta_0) < \delta_0^{\Omega(n)} \exp(2\Omega(n) \log \Omega(n))$$

if n is a prime power. The author was supported by the Foundation for Polish Science and by the Polish Research Committee (KBN grant No. 1P03A00826).

2. Proof of the lemma

It is easy to see that $a(n, \delta_0)$ depends only on δ_0 and the factorization type of n (the exponents in the factorization to prime powers). If p is a prime, $m = \Omega(n)$, $n = q_1 \cdots q_m$ with q_i primes (not necessarily distinct) and p_1, \dots, p_m distinct primes, then we show by induction on m that

$$a(p^m, \delta_0) \leq a(n, \delta_0) \leq a(p_1 \cdots p_m, \delta_0). \quad (1)$$

Indeed, this is trivial for $m = 1$ and if it is true for all $m' < m$, then

$$\begin{aligned} a(p^m, \delta_0) &= \delta_0 + \sum_{l=2}^m \frac{1}{l} \sum_{\substack{m_1+\dots+m_l=m \\ m_1, \dots, m_l \geq 1}} \dots \sum a(p^{m_1}, \delta_0) \dots a(p^{m_l}, \delta_0) \\ &\leq \delta_0 + \sum_{l=2}^m \frac{1}{l} \sum_{\substack{m_1+\dots+m_l=m \\ m_1, \dots, m_l \geq 1}} \dots \sum \prod_{i=1}^l a(q_{m_1+\dots+m_{i-1}+1} \dots q_{m_1+\dots+m_i}, \delta_0) \\ &\leq a(n, \delta_0). \end{aligned}$$

Similarly, if we map divisors of $p_1 \dots p_m$ onto those of n by $\sigma(p_{i_1} \dots p_{i_k}) = q_{i_1} \dots q_{i_k}$, then

$$\begin{aligned} a(n, \delta_0) &\leq \delta_0 + \sum_{l=2}^m \frac{1}{l} \sum_{\substack{n_1 \dots n_l = p_1 \dots p_m \\ n_1, \dots, n_l \geq 2}} \dots \sum a(\sigma(n_1), \delta_0) \dots a(\sigma(n_l), \delta_0) \\ &\leq a(p_1 \dots p_m, \delta_0). \end{aligned}$$

Thus (1) is proved. Further note that $a(n, \delta_0)$ is, for a fixed $n \geq 2$, a polynomial of degree $\Omega(n)$ in δ_0 , with nonnegative coefficients. Let $b(m)$ denote the leading term of $a(2^m, \delta_0)$ and $c(m) = a(p_1 \dots p_m, 1)$, with p_1, \dots, p_m distinct primes, $m \in \mathbb{N}$. We have:

$$\delta_0^{\Omega(n)} b(\Omega(n)) \leq a(n, \delta_0) \leq \delta_0^{\Omega(n)} c(\Omega(n)), \quad n \geq 2.$$

Moreover $c(1) = 1$ and

$$c(m) = 1 + \sum_{l=2}^m \frac{1}{l} \sum_{\substack{m_1+\dots+m_l=m \\ m_1, \dots, m_l \geq 1}} \dots \sum \frac{m!}{m_1! \dots m_l!} c(m_1) \dots c(m_l), \quad m \geq 2,$$

because, given m_1, \dots, m_l as above, we have $\frac{m!}{m_1! \dots m_l!}$ choices of n_1, \dots, n_l with $n_1 \dots n_l = p_1 \dots p_m$ and $\Omega(n_i) = m_i$, $i = 1, \dots, l$. Hence, for $m \geq 2$ we have

$$\begin{aligned} \frac{c(m)}{m!} &= \frac{1}{m!} + \sum_{m_1=1}^{m-2} \frac{c(m_1)}{m_1!} \left(\frac{c(m-m_1)}{2(m-m_1)!} \right. \\ &\quad \left. + \sum_{l \geq 3} \frac{l-1}{l} \cdot \frac{1}{l-1} \sum_{m_2+\dots+m_l=m-m_1} \dots \sum \frac{c(m_2) \dots c(m_l)}{m_2! \dots m_l!} \right) + \frac{1}{2} c(m-1) \\ &< \frac{3}{2} \sum_{m_1=1}^{m-1} \frac{c(m_1)}{m_1!} \frac{c(m-m_1)}{(m-m_1)!}. \end{aligned}$$

We note that $c(m) < m! \exp(2(m-1) \log m)$, $m \geq 1$. Indeed, it is true for $m = 1$ and if it is satisfied for all $m' < m$, then

$$\frac{c(m)}{m!} < \frac{3}{2}(m-1) \exp(2(m-2) \log m) < \exp(2(m-1) \log m).$$

The upper bound follows. The lower bound is obtained, in a very similar way, from

$$b(m) \geq \frac{7}{6} \sum_{m_1=1}^{m-2} b(m_1)b(m-m_1) + \frac{1}{2}b(m-1).$$

and direct computation of $b(m)$ for $m = 1, \dots, 4$. ■

References

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